Stat 212b: Topics in Deep Learning
Lecture 9

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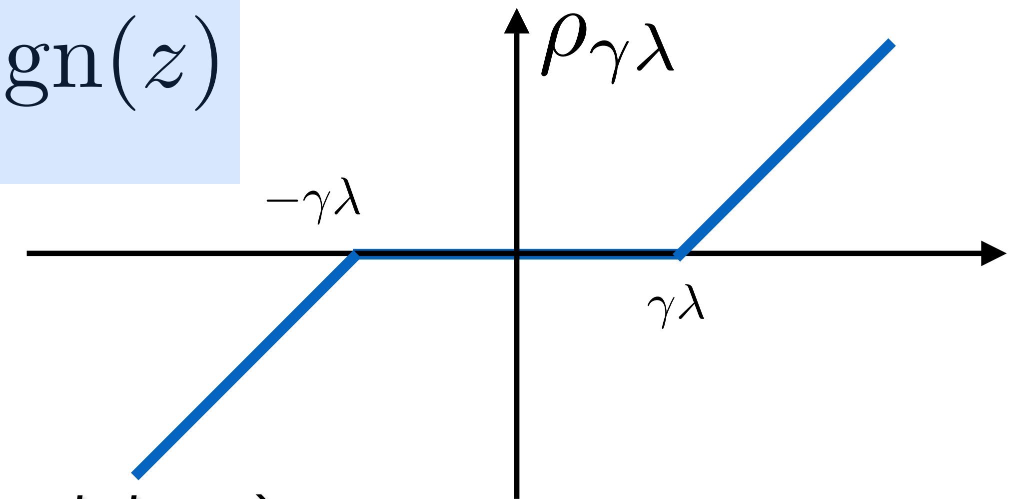
Review: Proximal Splitting and ISTA

- When $h_2(z) = \lambda \|z\|_1$, the proximal operator becomes

$$\text{prox}_{\gamma h_2}(z) = \max(0, |z| - \gamma\lambda) \cdot \text{sign}(z)$$

$\rho_{\gamma\lambda}$: *soft thresholding*



- ISTA algorithm (*iterative soft thresholding*):

$$z_{n+1} = \text{prox}_{\gamma_n h_2}(z_n - \gamma_n \nabla h_1(z_n))$$
$$\nabla h_1(z_n) = -D^T(x - Dz_n)$$

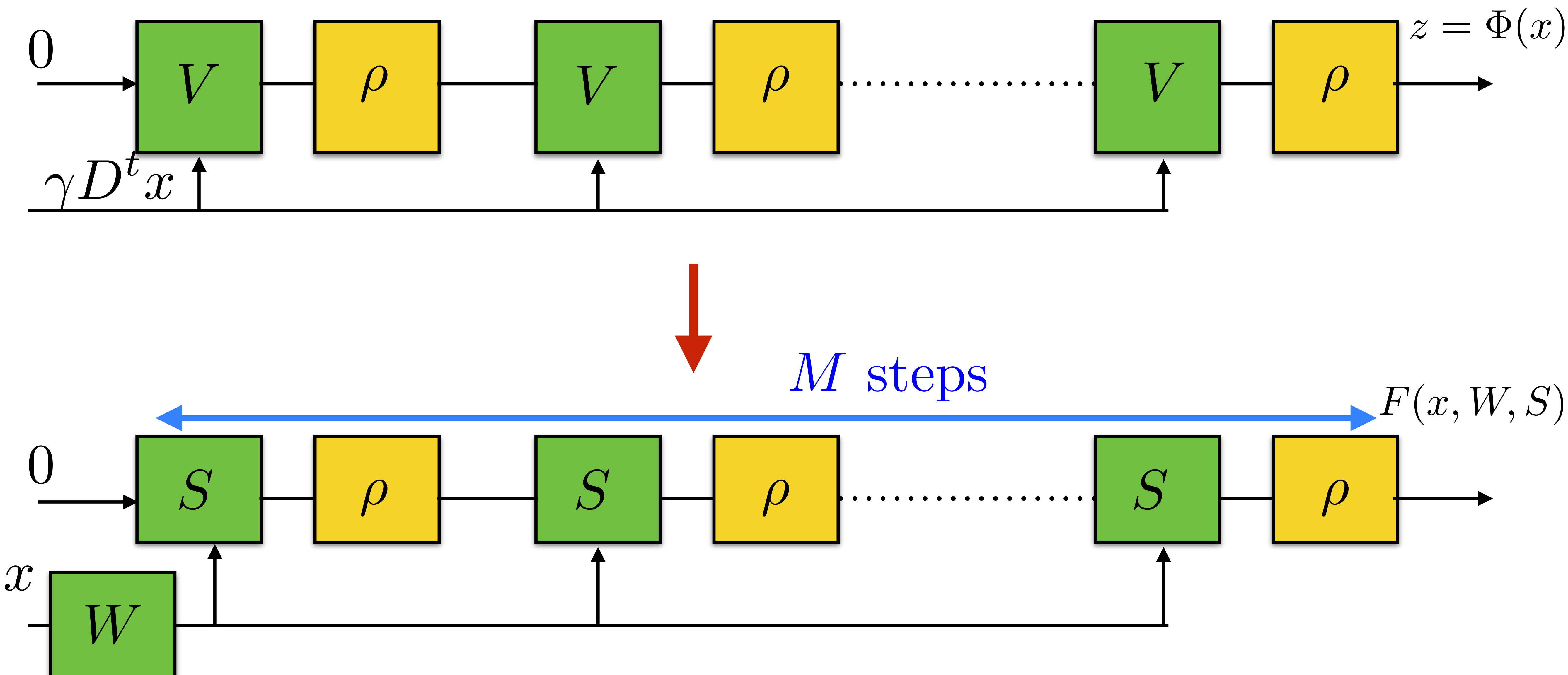
$$z_{n+1} = \rho_{\gamma\lambda}((\mathbf{1} - \gamma D^T D) z_n + \gamma D^T x)$$

  - converges in sublinear time $O(1/n)$ if $\gamma \in (0, 1/\|D^T D\|)$

- FISTA [*Beck and Teboulle,'09*]:
  - adds Nesterov momentum.
  - proven accelerated convergence $O(1/n^2)$

Review:From supervised Lasso to DNNs

- The Lasso (sparse coding operator) can be implemented as a specific deep network
- Can we accelerate the sparse inference with a shallower network, with trained parameters?



Objectives

- Random Forests and DNNs.
- Deformable Parts Model and CNNs.
- Structured Output Prediction
  - Graph Transformer Networks
  - CRFs and MRFs.
  - Examples: Detection, Segmentation, Pose Estimation.
- Embeddings
- Extensions to non-Euclidean domains
  - Spectral Networks
  - Spatial Transformer Networks

Review: Decision Trees

- Let $x = \{(x_1, y_1), \ldots, (x_T, y_T)\}$ be the input data, with $x_i \in \mathbb{R}^N$.
- Each node of the three selects a variable and splits using a threshold.

$$\Omega_i^j \subset \mathbb{R}^N \rightarrow (\Omega_{i+1}^l, \Omega_{i+1}^{l+1})$$
$$\Omega_i^j = \Omega_{i+1}^l \cup \Omega_{i+1}^{l+1} \;,\; \emptyset = \Omega_{i+1}^l \cap \Omega_{i+1}^{l+1}$$

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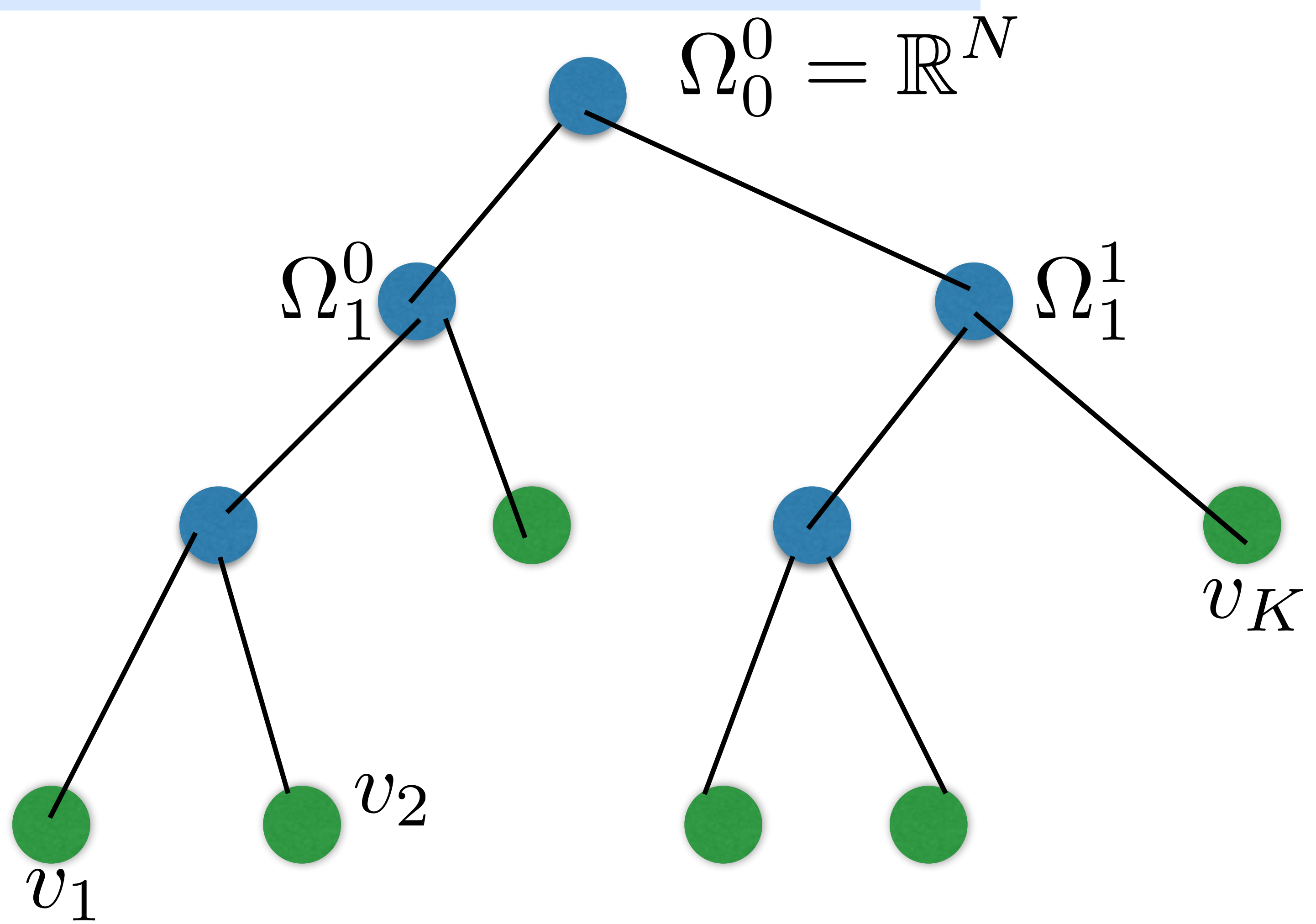
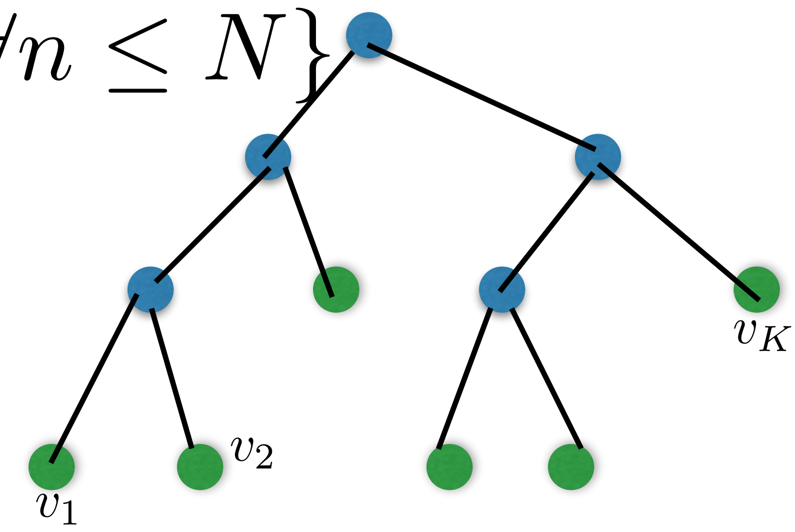


Decision Trees

The leaves $\{v_k\}$ of the tree define a partition of the input space into cubic sections:

$$\Omega_\infty^k = \{x \in \mathbb{R}^N \, ; \, \alpha_{k,n} \leq x_n \leq \beta_{k,n} \; \forall n \leq N\}$$



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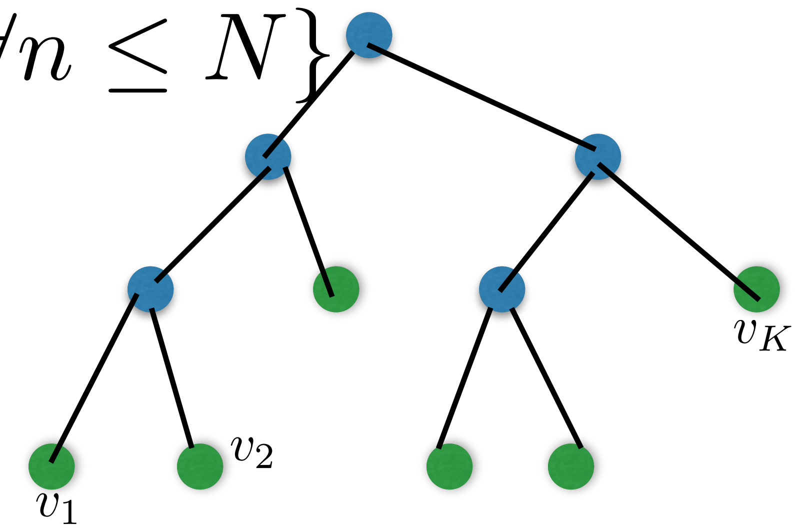
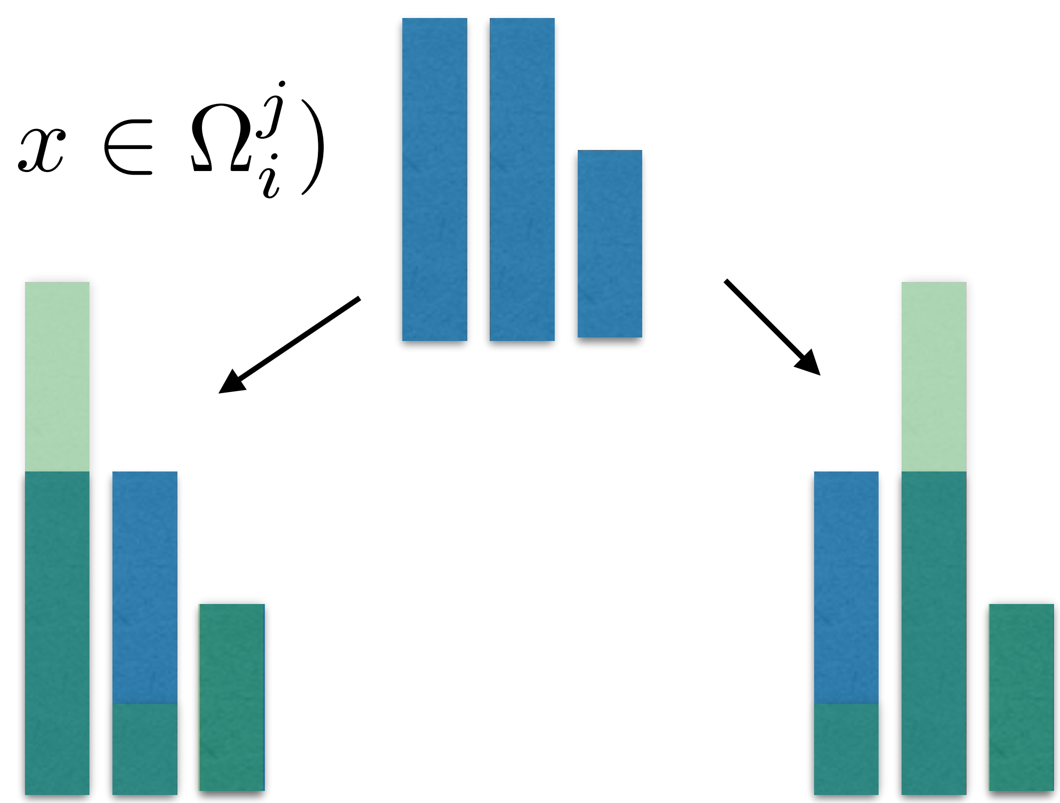


- Each split optimizes the entropy in the label distribution:

$p(y \mid x \in \Omega_i^j)$



$p(y \mid x \in \Omega_{i+1}^l)$

$p(y \mid x \in \Omega_{i+1}^{l+1})$

Random Forests

- A decision tree can capture interactions between different variables, but it is very noisy (ie unstable).
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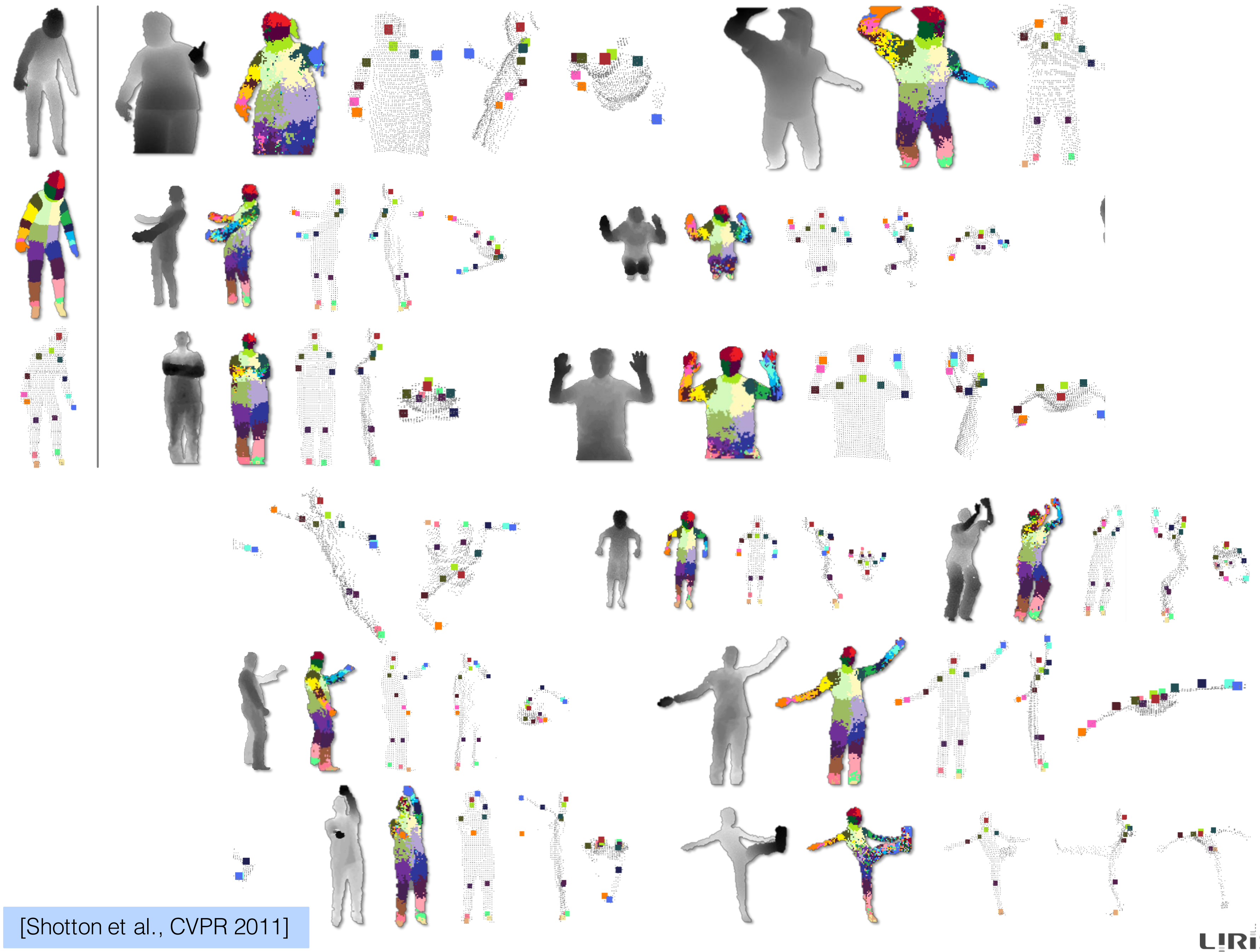
- Evaluation and training are extremely efficient.
- By appropriately introducing *randomization*, we can construct an ensemble of random trees: the so-called *random forests*.

- We draw bootstrapped samples of the training set, and each split in the tree is calculated *only on a small random subset of variables* (typically of size $O(\sqrt{N})$).
- The prediction is the aggregate prediction (ie voting) of each tree.

Random Forests

- Successful across a wide range of classification and regression problems.



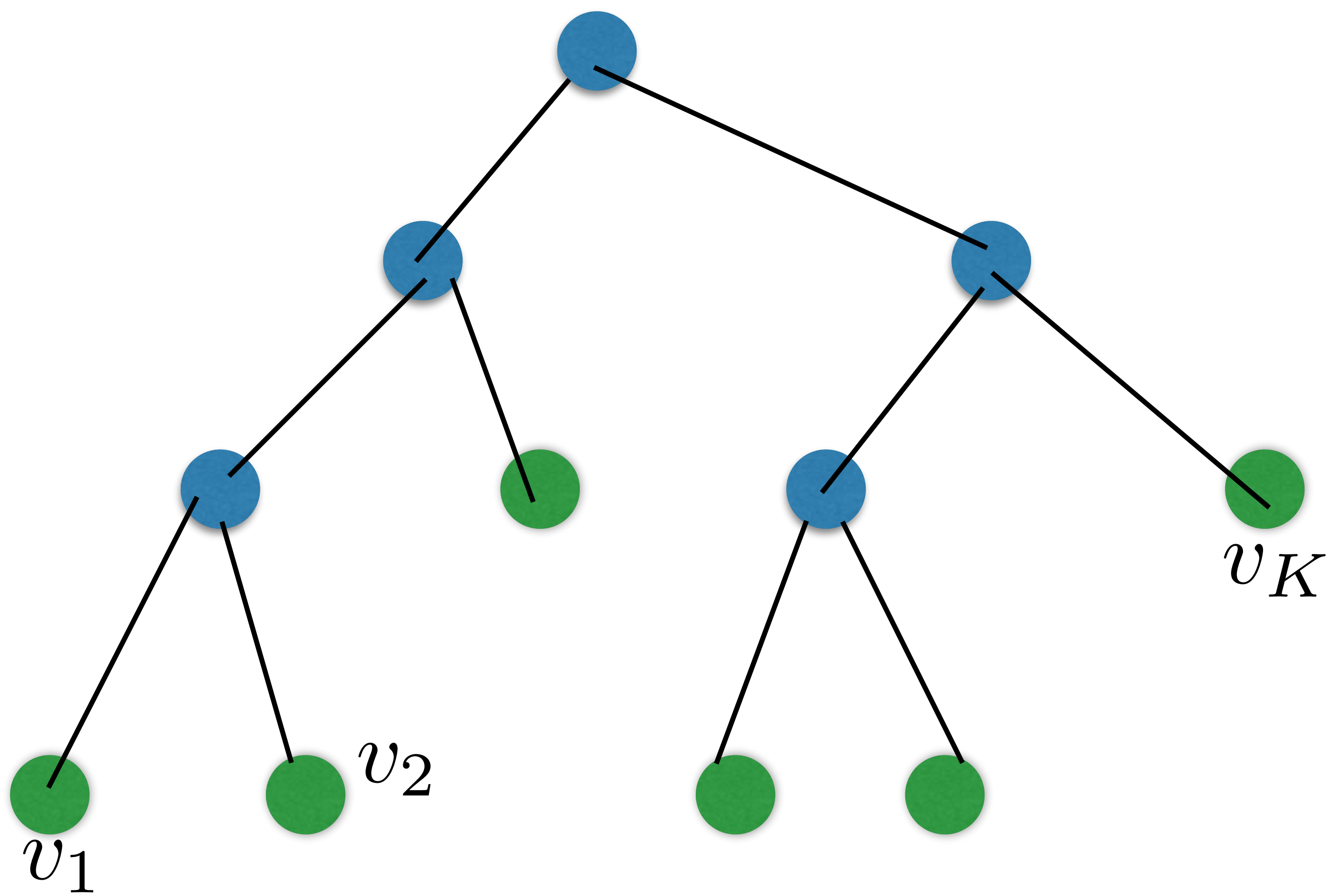
Real-Time Pose Estimation from Kinect measurements (CVPR'11)

LIRIS

(figure from Ch. Wolf slides)

Embed RF into Deep Neural Networks?

- Q: How to write a Decision Tree in terms of a network?



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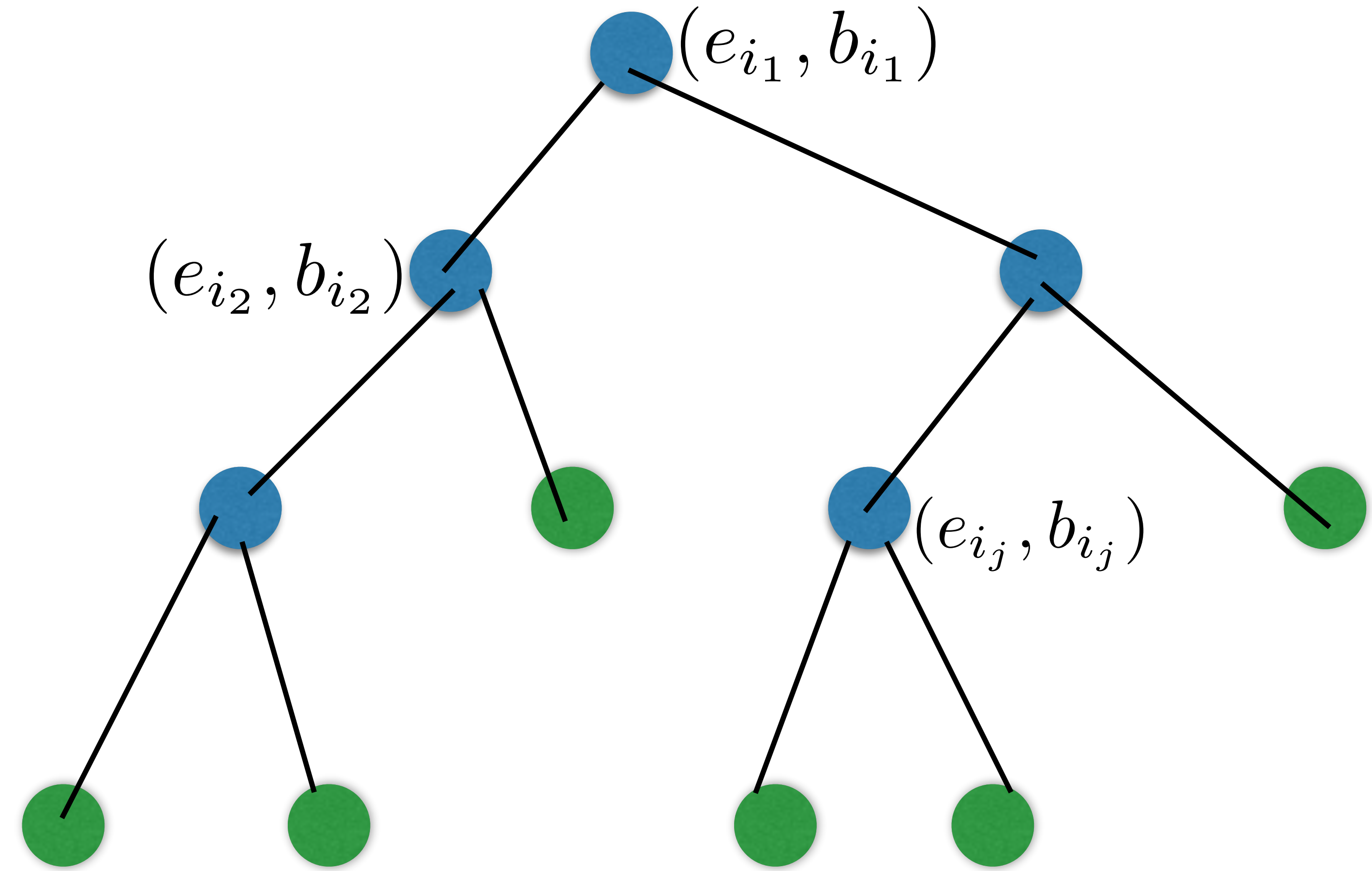
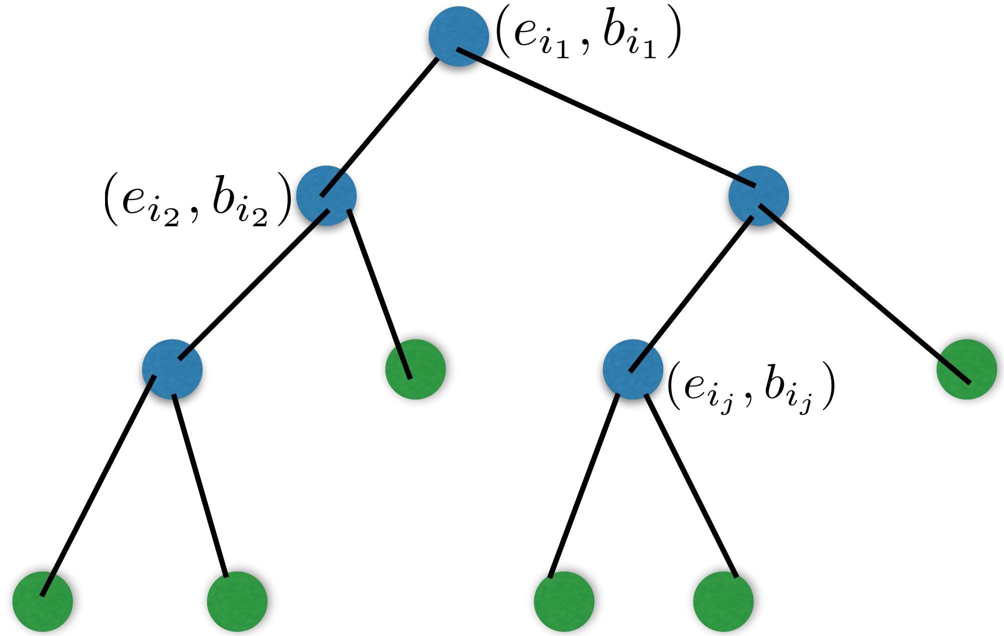


- Each node in the tree amounts to a comparison of the form $\langle x, e_{i_j} \rangle \leq b_{i_j}$

Embed RF into Deep Neural Networks?



- Let $W = (e_{i_1}, e_{i_2}, \ldots, e_{i_S}) \in \mathbb{R}^{S \times N}$ (S: number of nodes)
- Let $b = (b_{i_1}, b_{i_2}, \ldots, b_{i_S}) \in \mathbb{R}^{S}$ (L: number of leaves)

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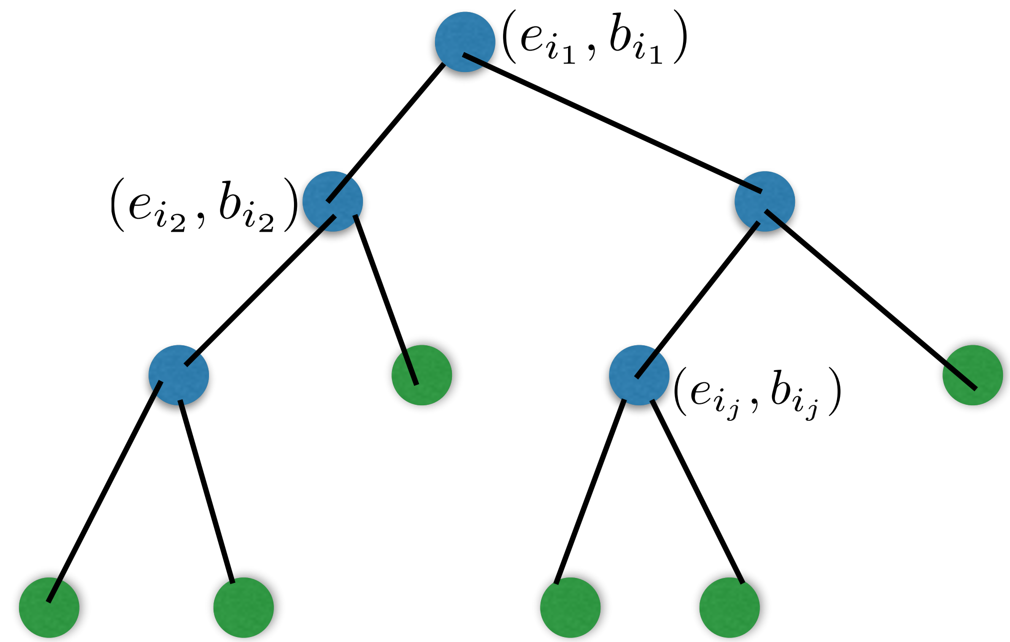

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- Let $y = \text{sign}(Wx + b)$
- For each leaf l, let $v_l = (\pm 1, \ldots, \pm 1)$ encoding the tree path, and $V = (v_1, \ldots, v_l) \in \mathbb{R}^{L \times S}$.

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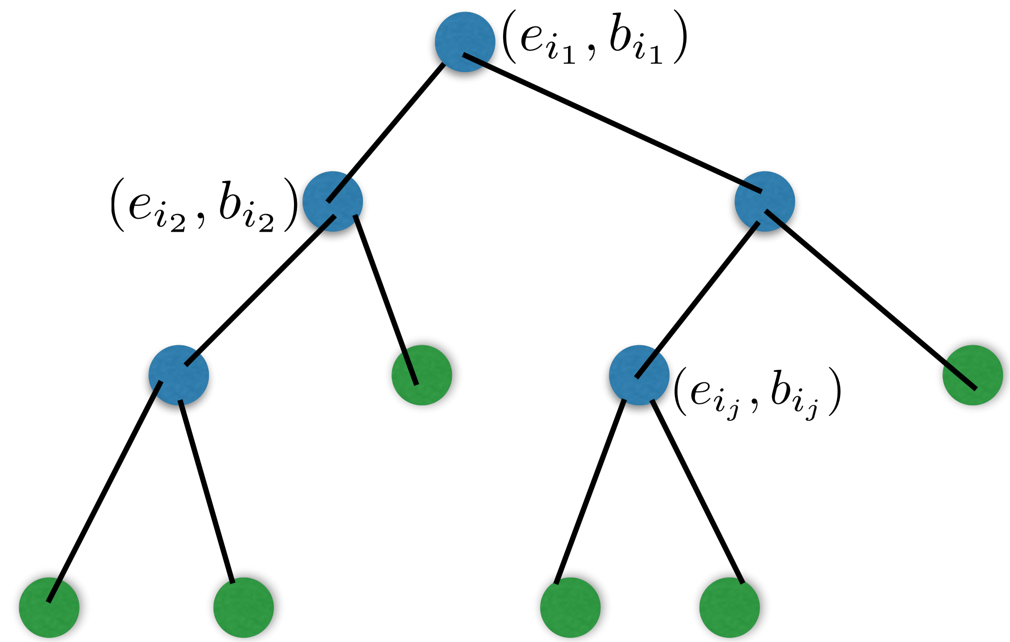

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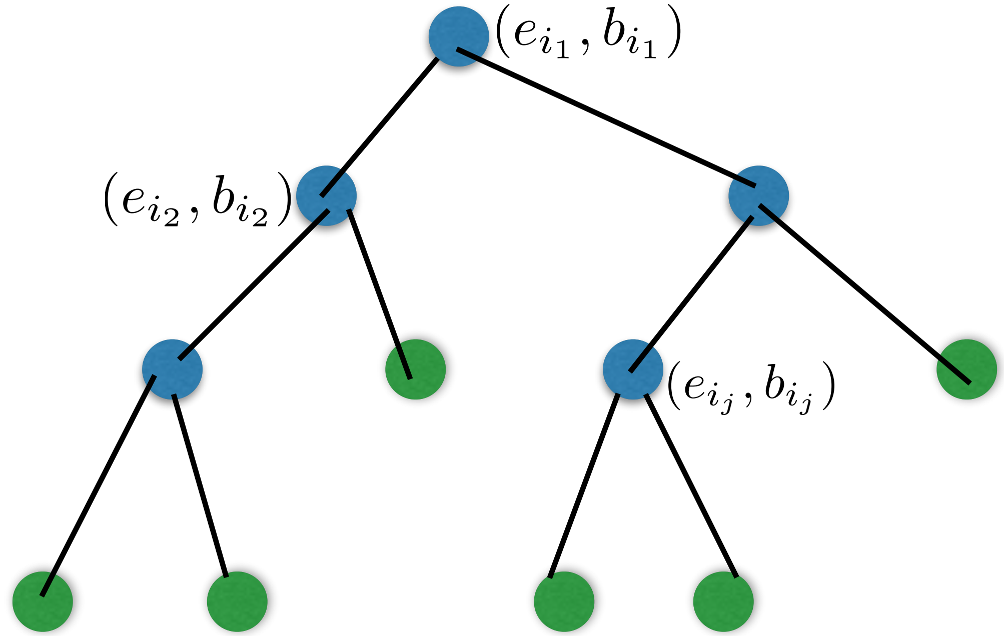

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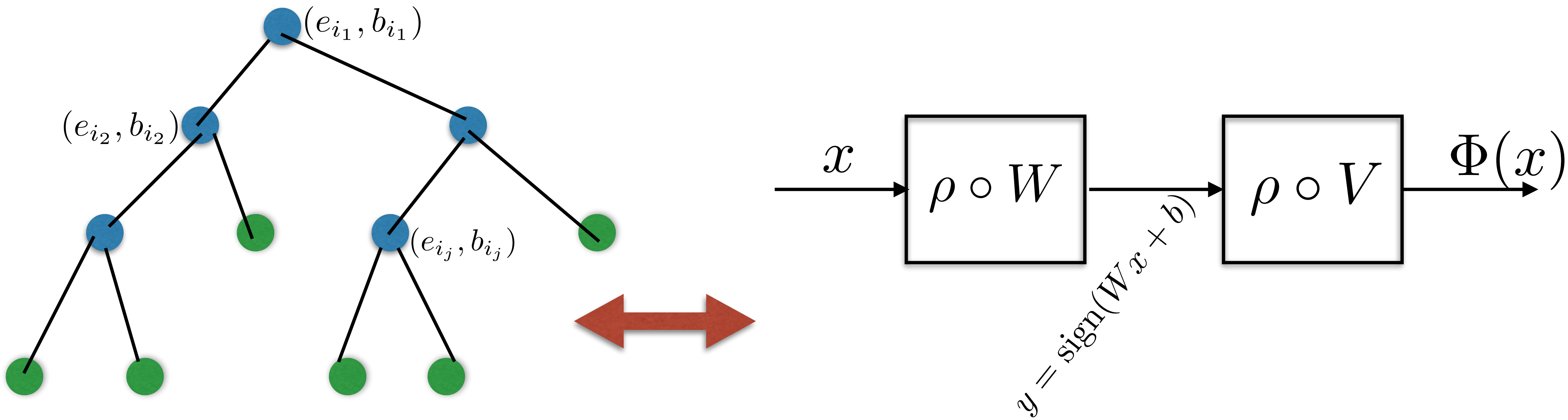
Let $\Phi(x) = \text{sign}(Vy - \tilde{b})$

Embed RF into Deep Neural Networks?



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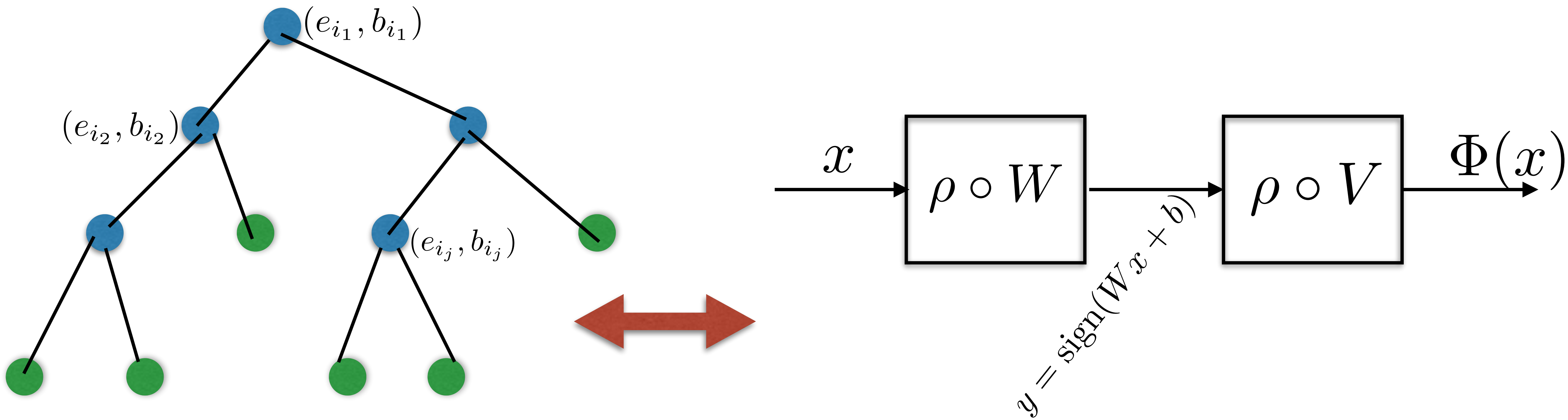
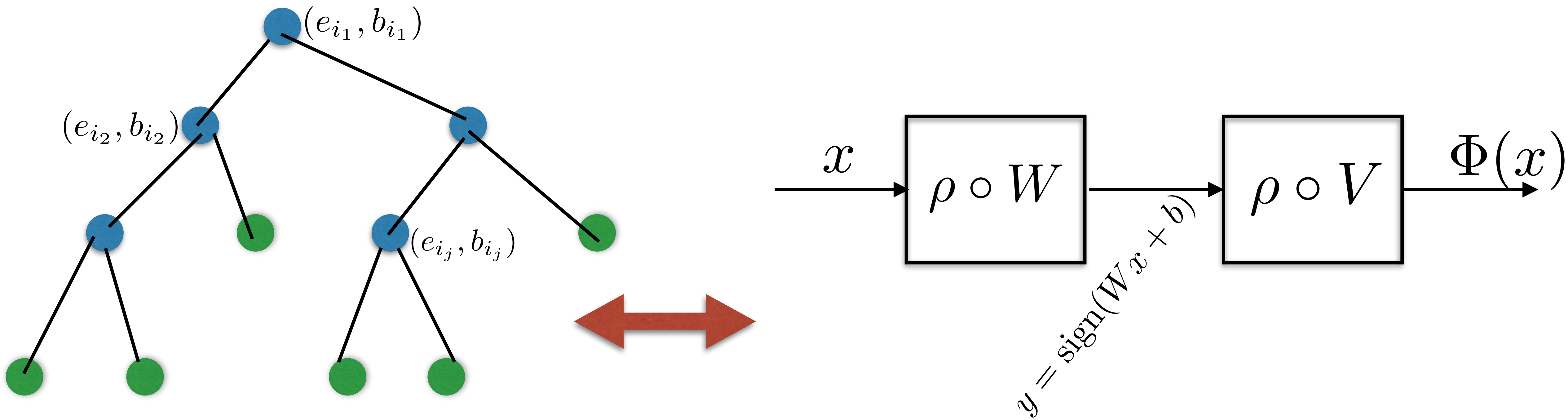


- $\Phi(x) \in \mathbb{R}^L$ is a *one-hot* vector encoding $x \in \Omega_\infty^l$, $l \leq L$.
  $(x \in \Omega_\infty^l \Leftrightarrow \Phi(x)_l = 1, \Phi(x)_k = 0\,, k \neq l)$

- A decision tree can thus be thought as a special two-layer network.

Embed RF into Deep Neural Networks?



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- The Random Forest is obtained with an ensemble of two-layer networks.
- Training is radically different: greedy in RF versus gradient descent in Deep Learning.

Random Forests and CNNs

- Random Forests thus also consider piecewise linear regions of the input space.

- However the encoding of these regions is different from that of a deep ReLU network.

- Computationally more efficient
- No gradient descent training
- Less expressive

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- However the encoding of these regions is different from that of a deep ReLU network.

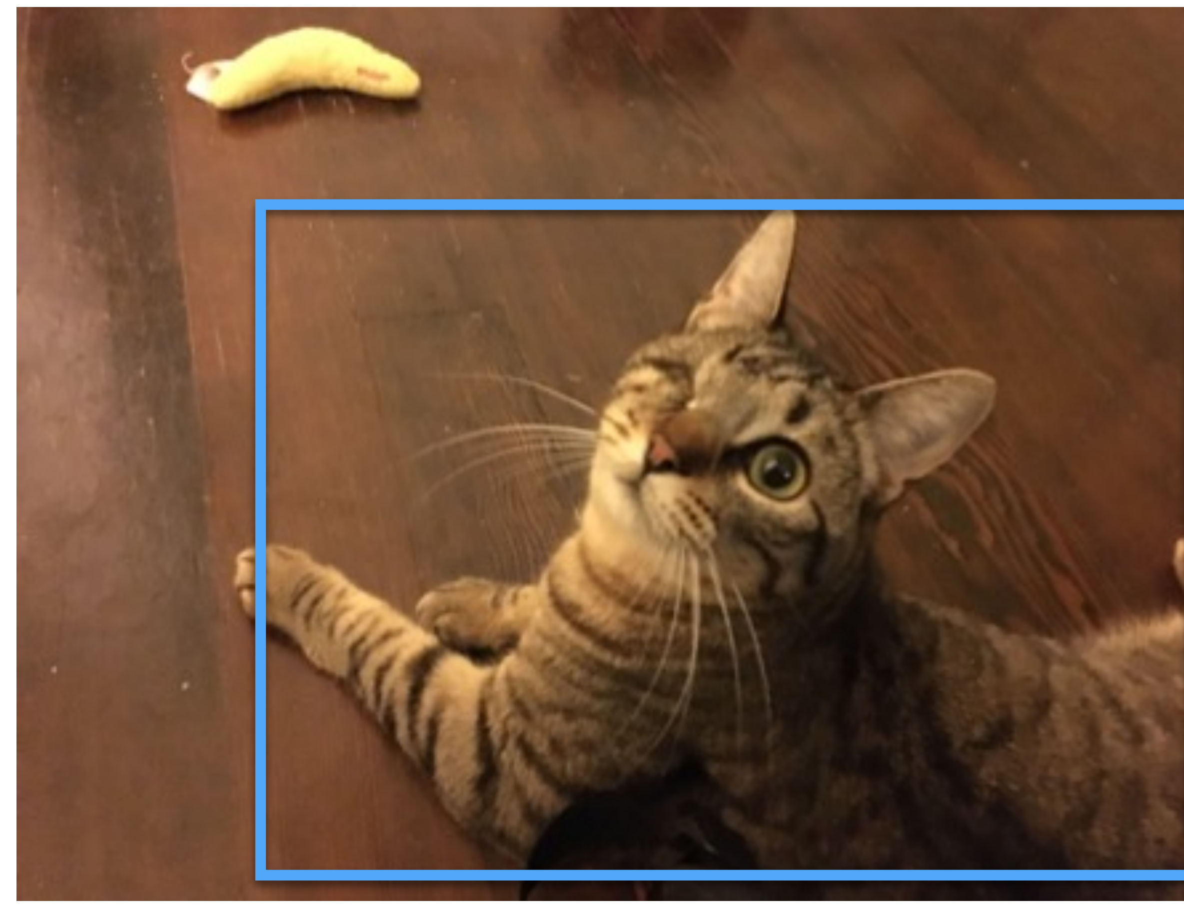
- Computationally more efficient
- No gradient descent training
- Less expressive
- One can also combine both models (eg "Deep Neural Decision Forests", [Kontscheider et al, MSR,'15]).

Beyond Object Classification

classification

"cat"

classification and localization



"cat"

single object problems

Beyond Object Classification

object detection

{"bicycle", "man", "woman", "house", "house", "house"}

semantic segmentation

multiple object problems

Deformable Parts Model

[Fischler & Elschlager '73, Felzenszwalb & Huttenlocher '00]

- It is a graphical model with two key components:
  - *Parts:* local structures or templates.
  - *Springs:* Connections between parts encoding our geometric prior.

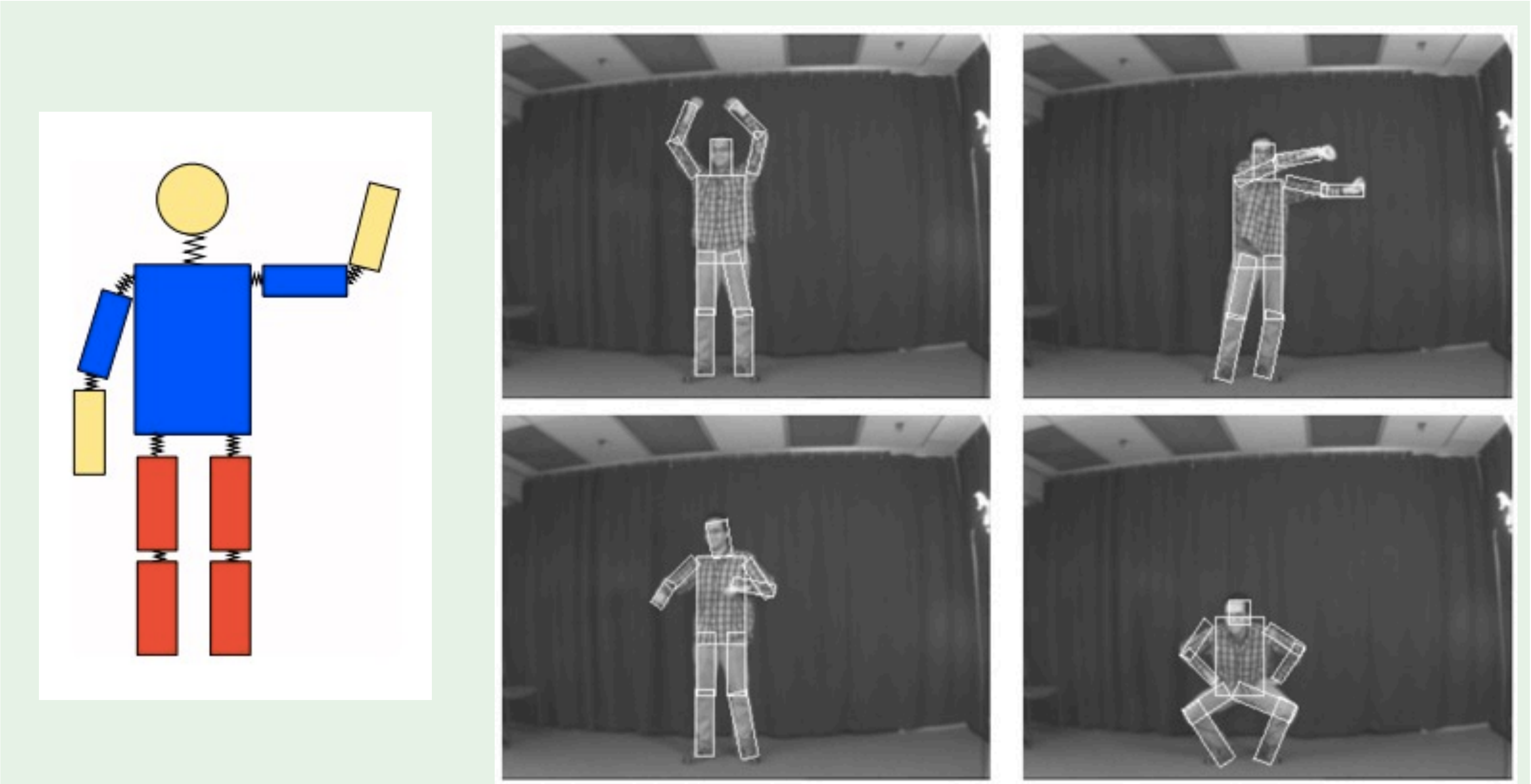
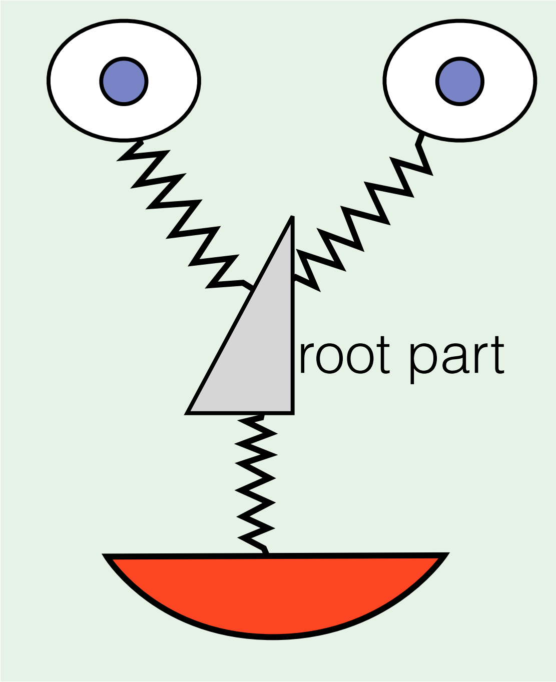
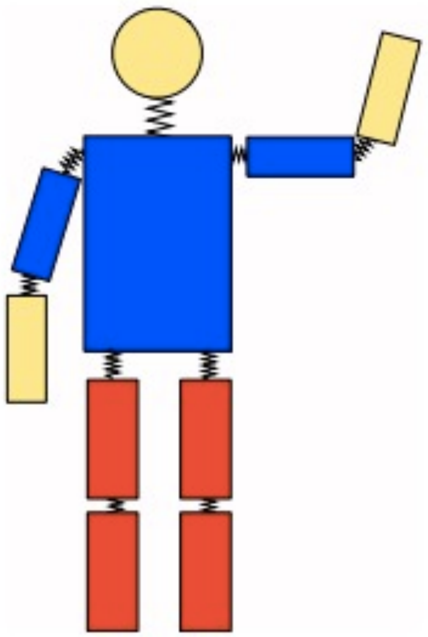


Image: [Felzenszwalb and Huttenlocher 05]

(figure credit: Ross Girshick)

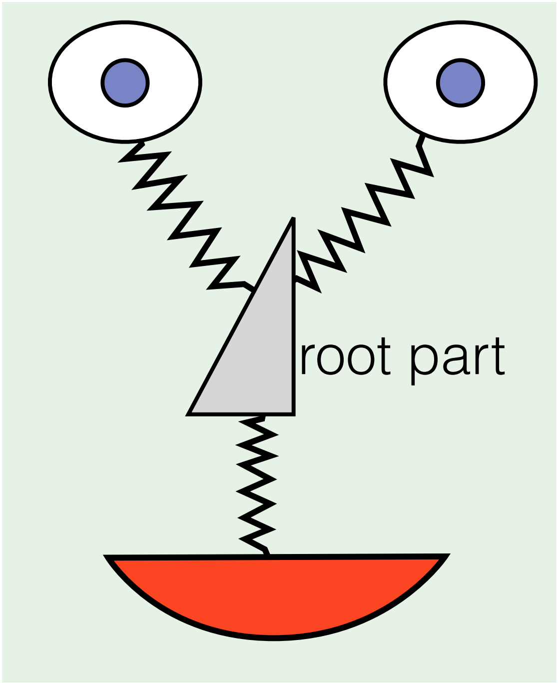
Deformable Parts Model

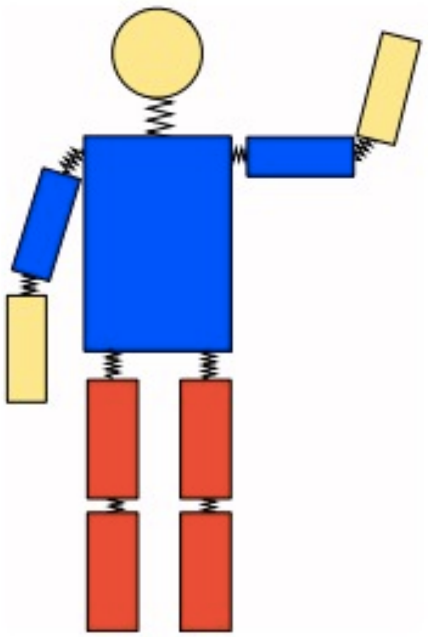




- Intuition:
  - Modeling each subpart is easier than modeling whole objects because they are shared across different instances.

Deformable Parts Model

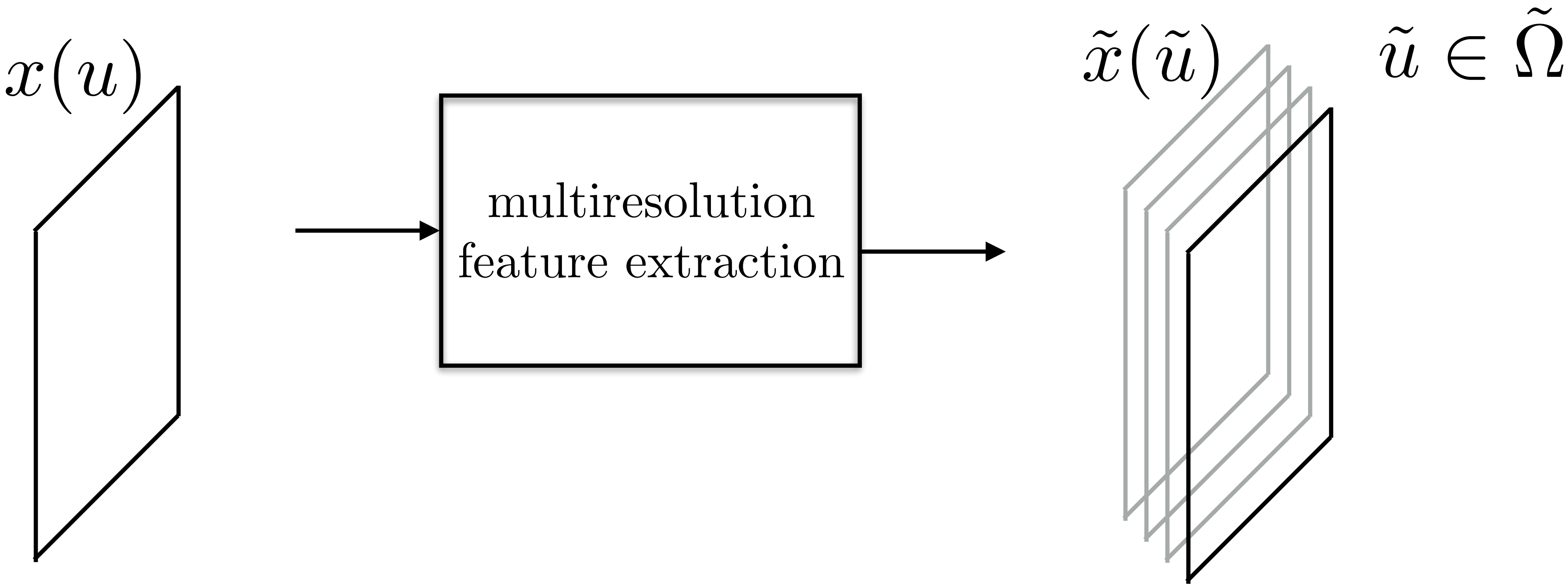




- Intuition:
  - Modeling each subpart is easier than modeling whole objects because they are shared across different instances.
  - The model also needs to capture the typical deformation between parts.
  - Parts can be either localized in space or global if extracted from low-frequency measurements (MultiResolution Analysis such as Laplacian Pyramid).

Deformable Parts Model

- Typically we start by extracting multi-resolution features, either by a predefined transform (such as SIFT, HoG, Scattering) or using CNN features:



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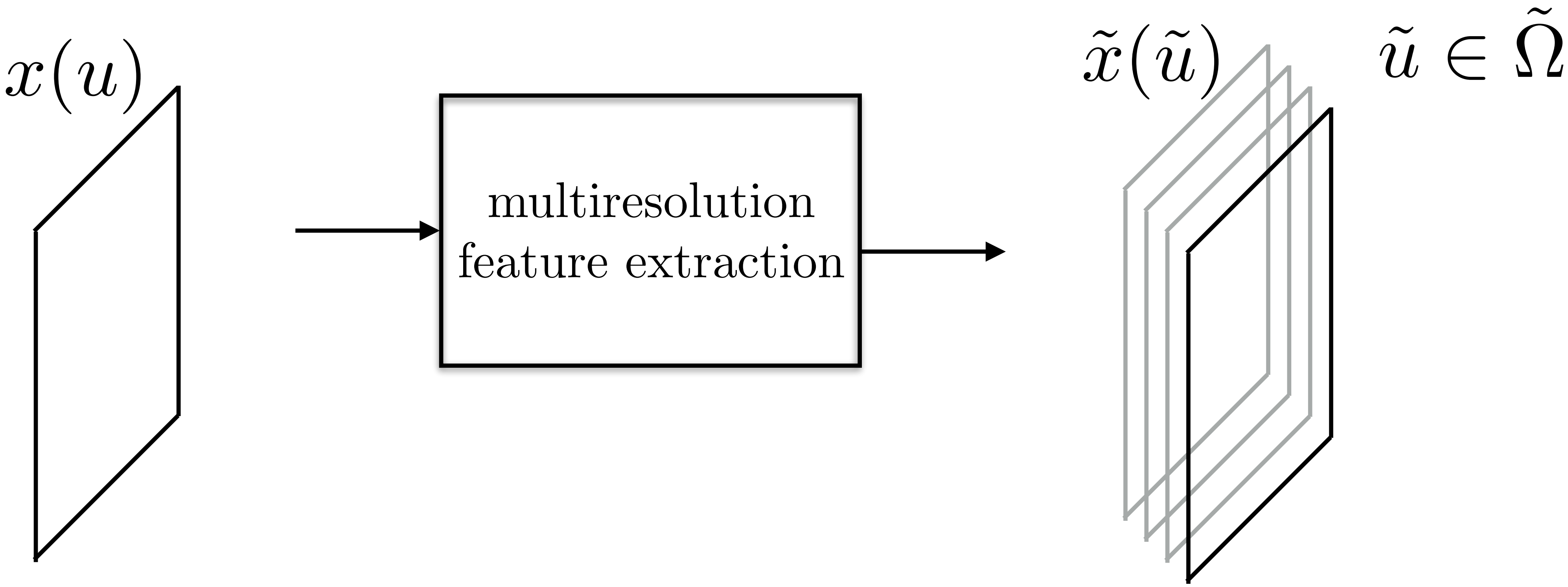

- Deformations on coordinates $\tilde{u}$ model more general transformations (rotations, dilations, appearance, etc.)

Deformable Parts Model

- A DP model for an object with n parts is given by a $(n+2)$-tuple $(F_0, P_1, \ldots, P_n, b)$ where:
  - F_0: *root* filter.
  - b: bias term
  - $P_i = (F_i, v_i, d_i)$ part model, where F_i is the filter for part i, v_i the *anchor* and d_i specifies deformation cost wrt anchor.

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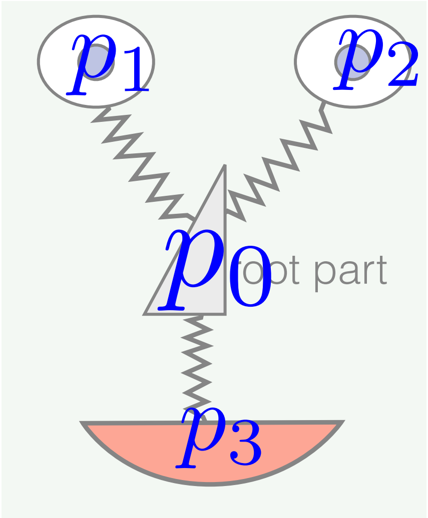
- An object hypothesis specifies the locations of root and parts:

$$z = (p_0, \ldots, p_n) \ , \ p_i \in \tilde{\Omega}$$



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- The *score* of an object hypothesis is

$$E(z) = \sum_{i=0}^{n} (F_i \star \tilde{x})(p_i) - \sum_{i=1}^{n} d_i^T \phi(p_i - (p_0 + v_i)) + b\ .$$

$\phi(\tau)$: deformation features (typically first two moments $|\tau|_j$, $|\tau|_j^2$)

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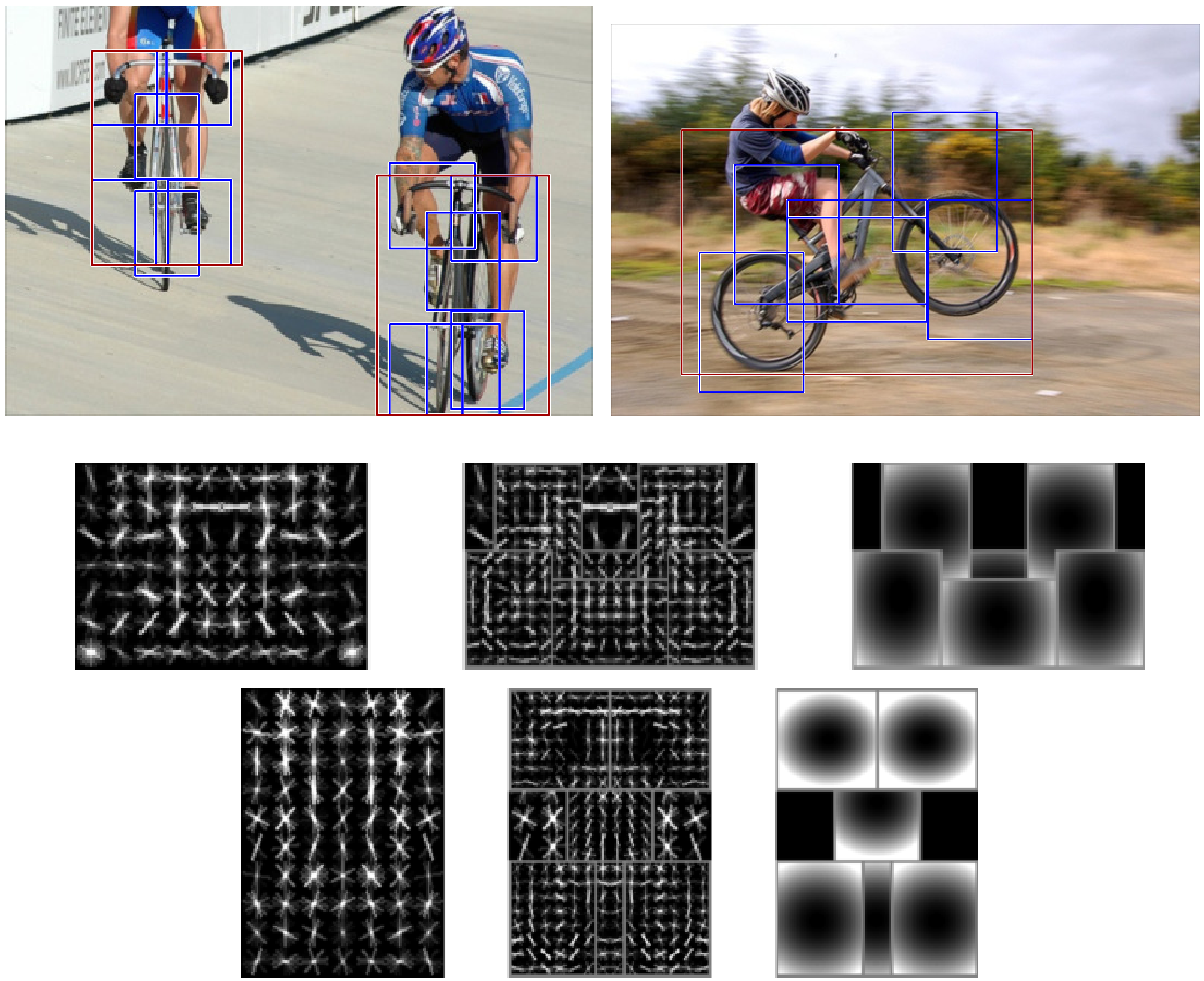
$\phi(\tau)$: deformation features (typically first two moments $|\tau|_j$, $|\tau|_j^2$)

- Efficient implementation using dynamic programming.
- Extension to include mixture models for objects.
- Trained with *Latent* SVM.

Deformable Parts Model

"Object Detection with discriminatively trained Deformable Parts Model", Felzenszwalb, Girshick et al. PAMI'10



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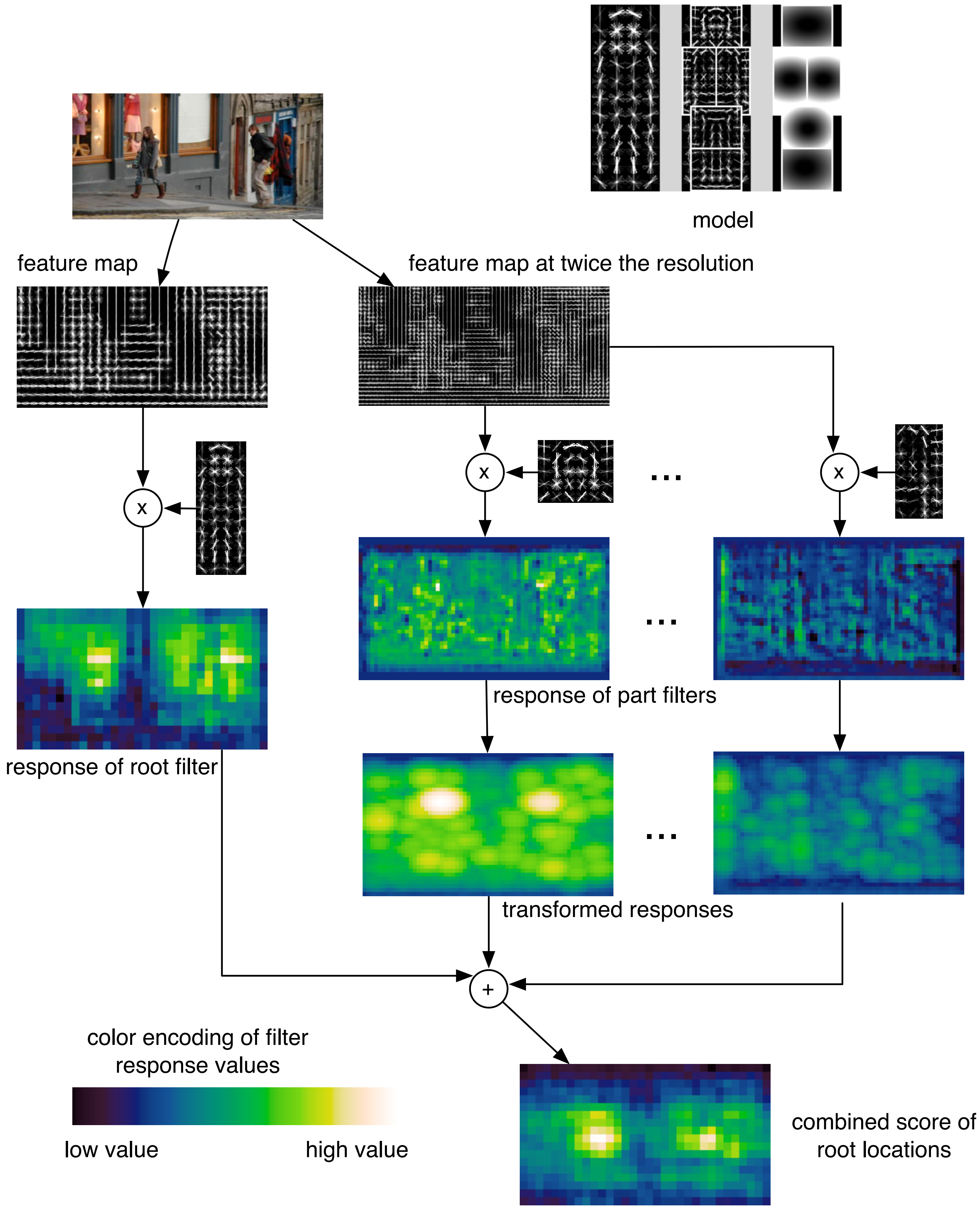

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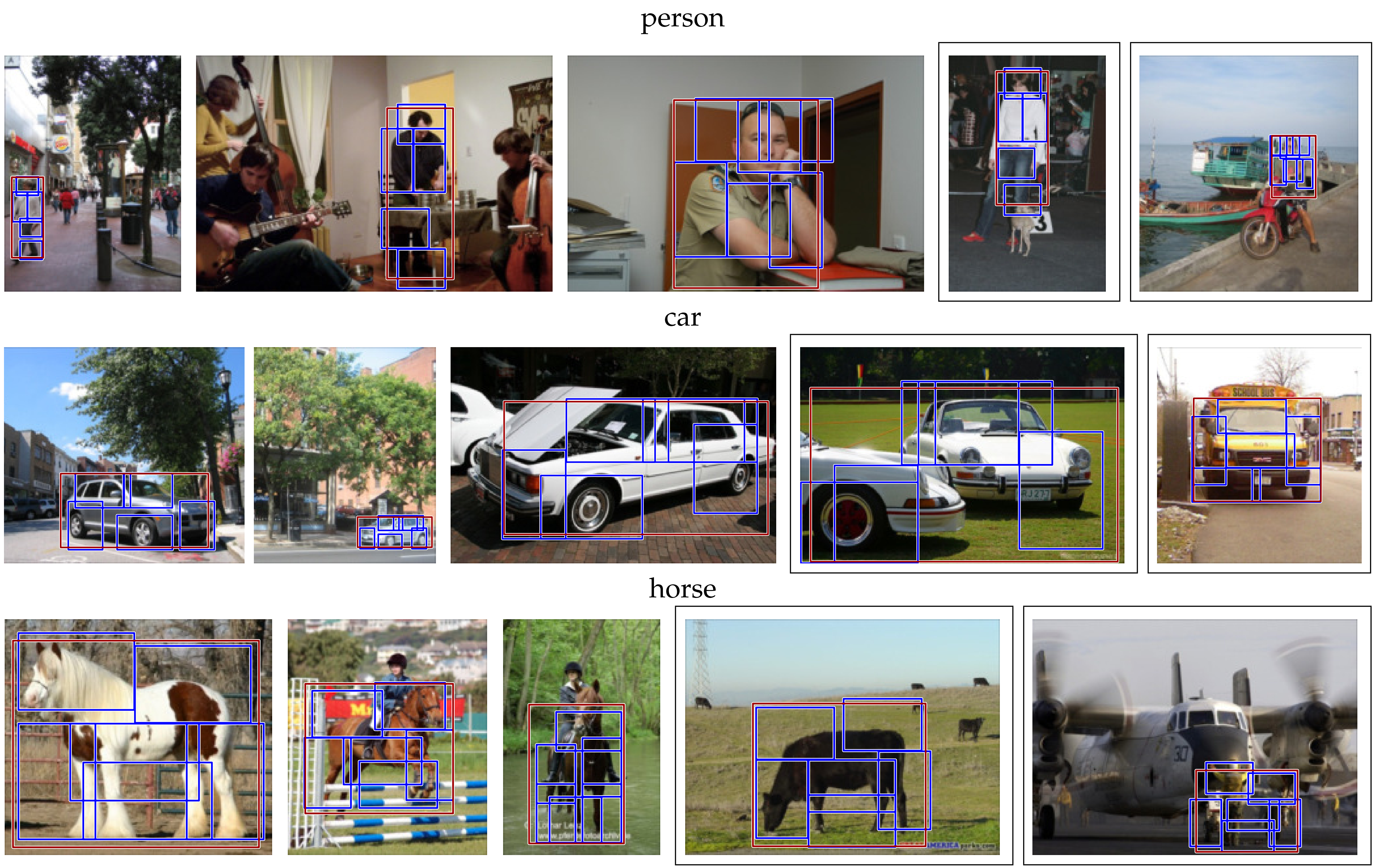
Deformable Parts Model



- State-of-the-art in PASCAL object detection pre-2012.
- Model contains "convolutions with penalized offsets".
- Can we relate it to generic CNNs?

DPM and CNNs

- The optimization of part offsets with respect to the anchor is a *distance transform*:

The distance transform of $x : \Omega \to \mathbb{R}$ is a function $D_x : \Omega \to \mathbb{R}$ defined by $D_x(u) = \max_q x(q) - d(u - q)$

DPM: $d(r) = r^T A r + b r$ quadratic form

Max-Pooling: $d(r) = \begin{cases} 0 & \text{if } |r| \leq K \ , \\ \infty & \text{otherwise } . \end{cases}$

DPM and CNNs

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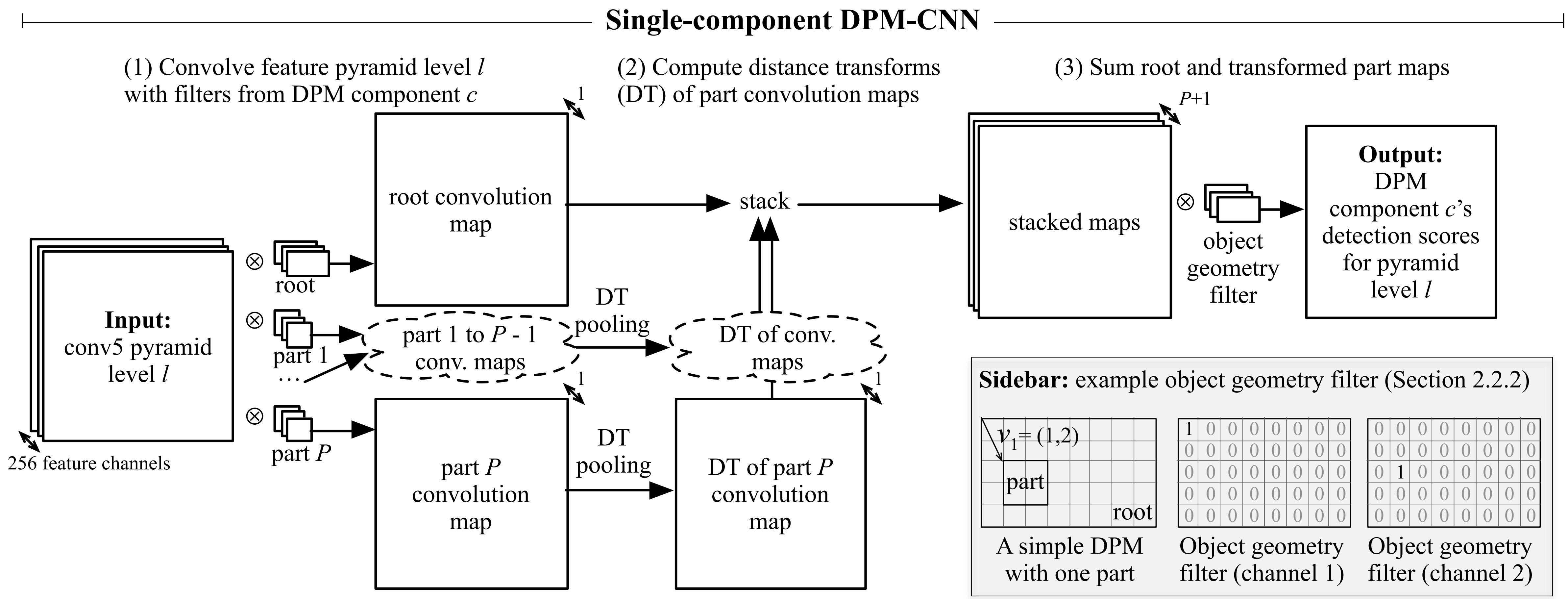
$$\text{Max-Pooling: } d(r) = \begin{cases} 0 & \text{if } |r| \leq K \ , \\ \infty & \text{otherwise} \ . \end{cases}$$

- Therefore, one can train a DPM end-to-end as a particular instance of a CNN.

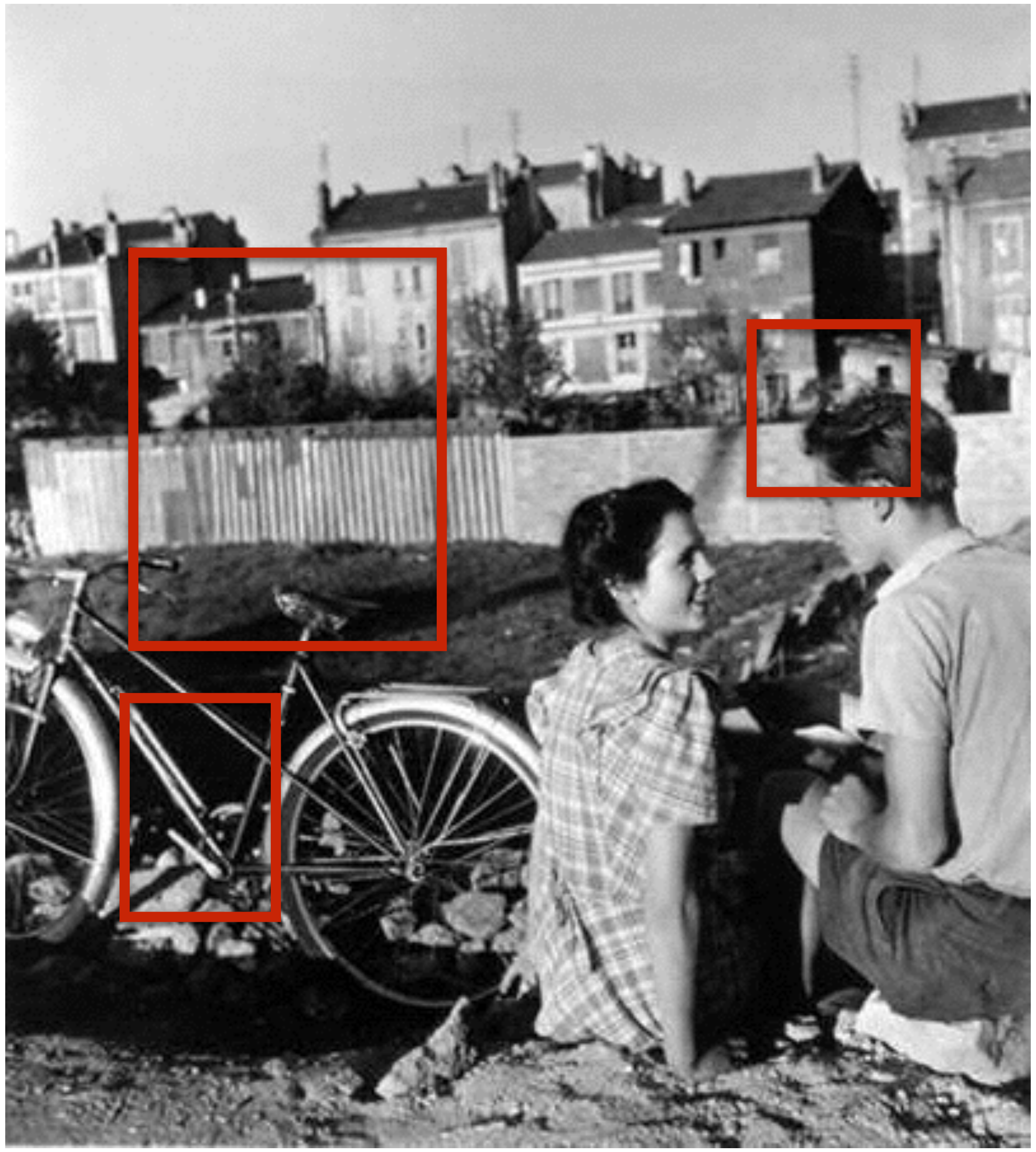
DPMs and CNNs



[Girshick et al, CVPR'15]

- Large improvement from using handcrafted features.
- State-of-the-art amongst "*sliding windows*" methods, i.e. translation invariant.

Region-based CNN (R-CNN)



- Suppose that for each bounding box we ask: is there a {house, bicycle, dog, man, …, *none*} ?

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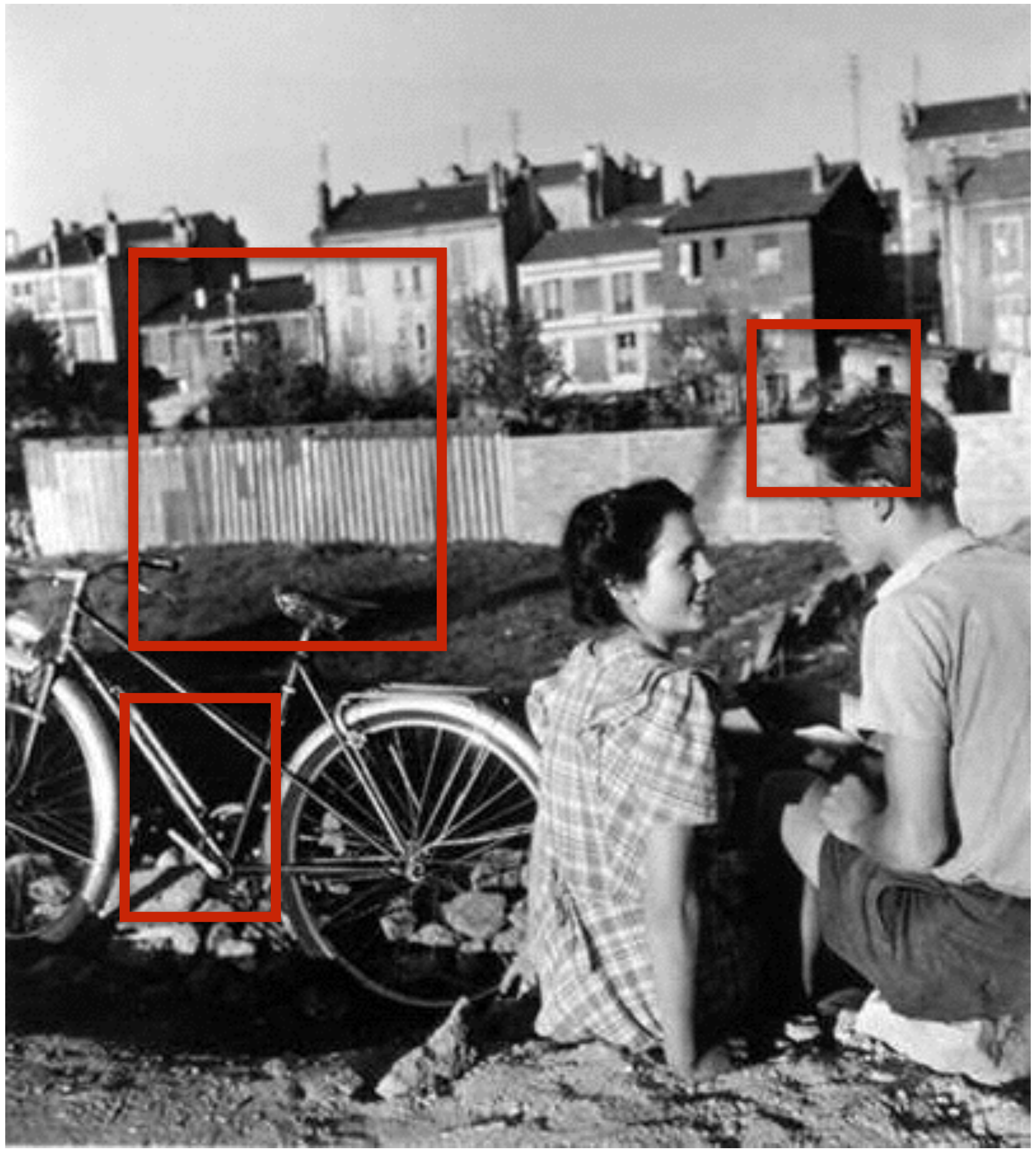
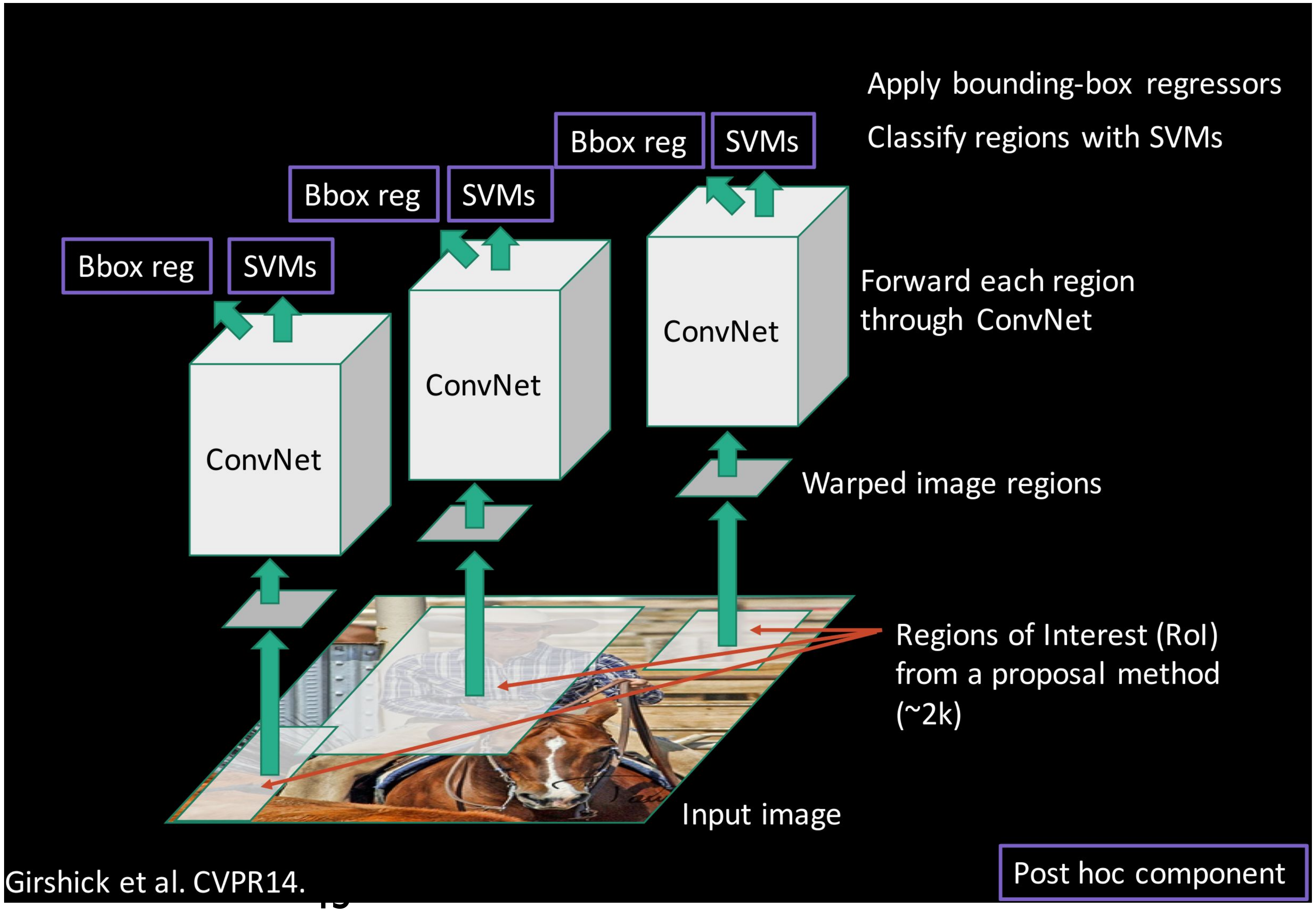


- Suppose that for each bounding box we ask: is there a {house, bicycle, dog, man, …, *none*} ?
- This is standard object classification.

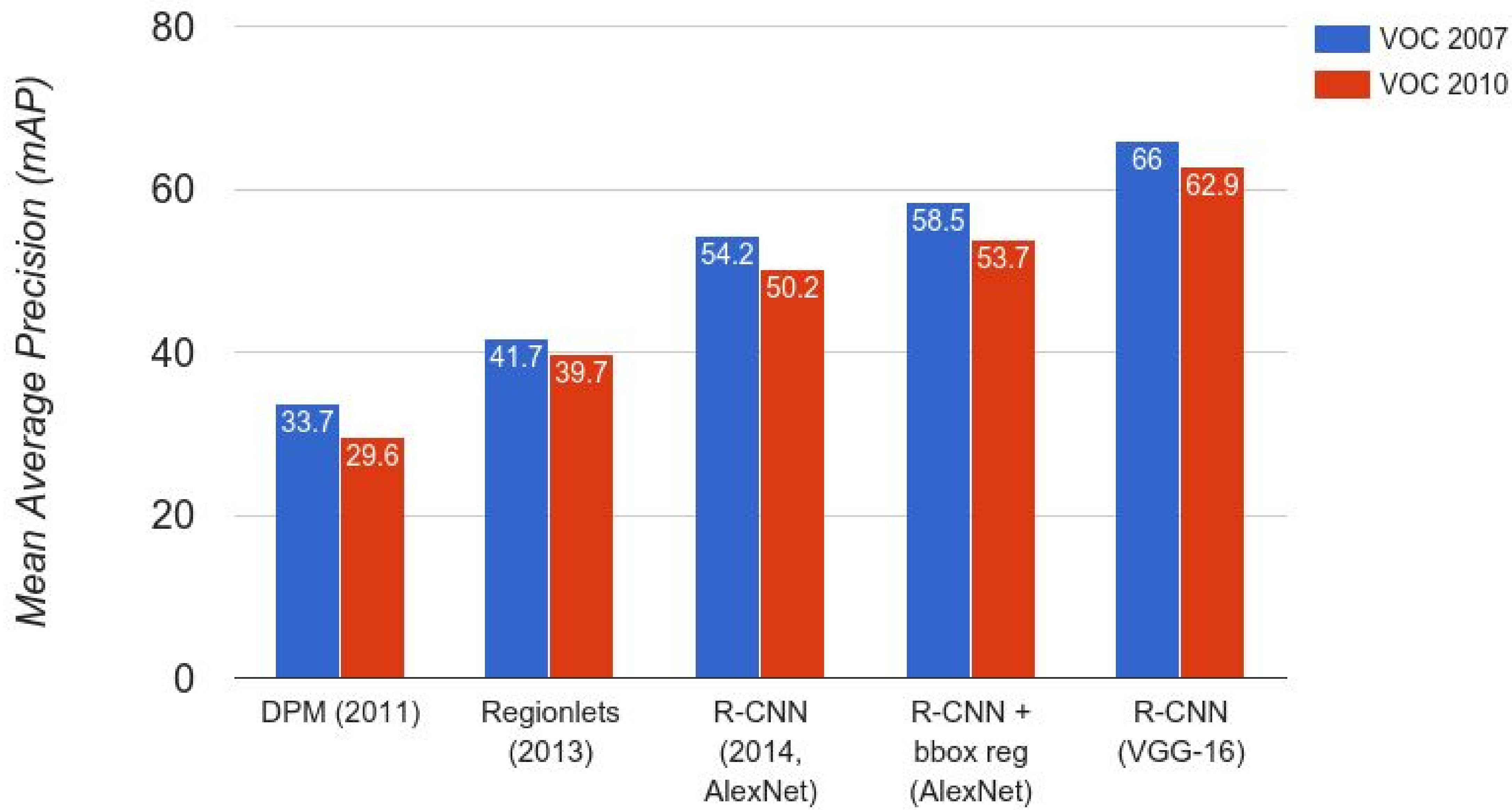
R-CNN [R. Girshick et al, 14-15]

- Rather than testing every possible rectangular region, we rely on a Region Proposal algorithm (which can also be done by a CNN).
- Each proposal region is warped and analyzed with another CNN.



R-CNN [R. Girshick et al, 14-15]

- Several improvements relating speed and performance (Fast R-CNN, Faster R-CNN) and replacing pre-trained CNN architectures (ResNet).



(figure credit: Stanford CS-231n lecture 8)

Structured Output Prediction

- Standard classification is only concerned with estimating conditional probabilities of the form $p(y \mid x)$:

$$x \in \mathcal{X} \longrightarrow y \in \mathcal{Y} \qquad \mathcal{Y} = \{s_1, \ldots, s_L\} \text{ (classification)}$$

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$\mu(y)$ models the output unconditional probability.

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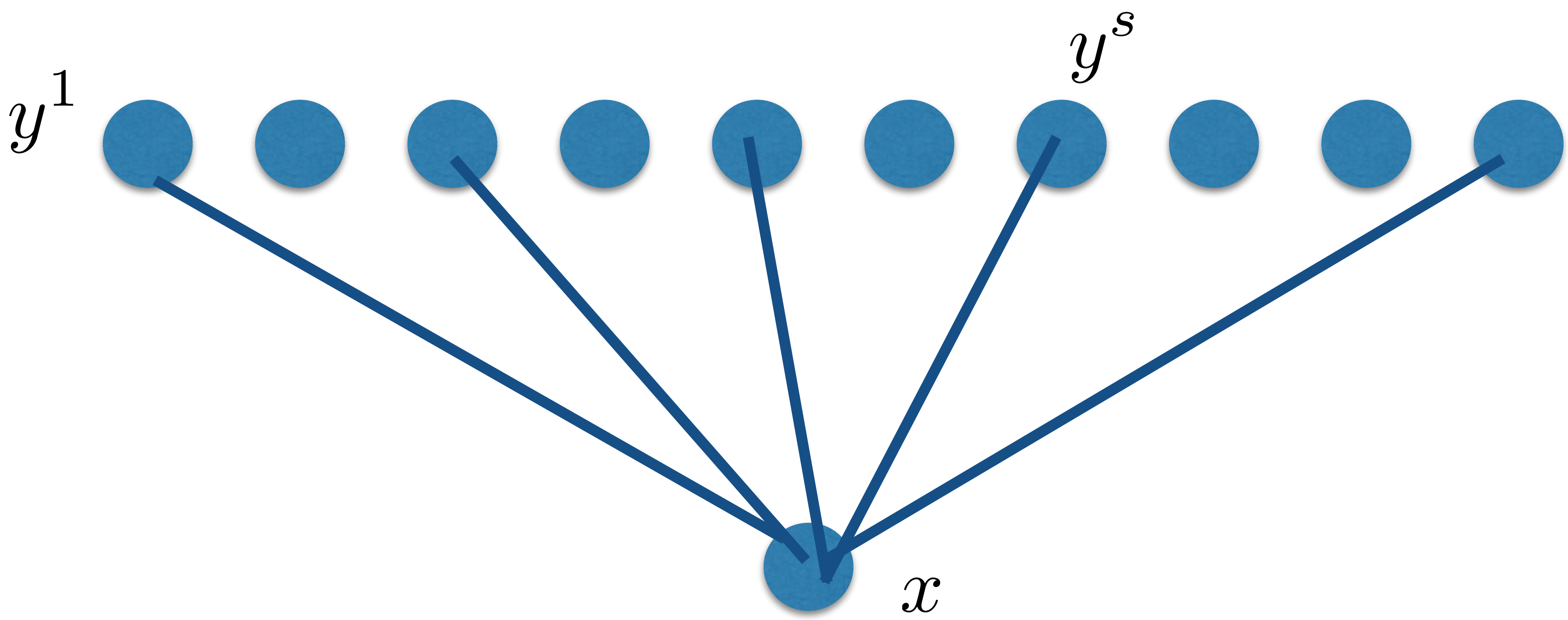
- Q: How to regularize the estimation of $p(y \mid x)$ with $\mu(y)$

Structured Output Prediction

- Examples:
  - Natural Language Processing: Translation, Summarization, Question Answering.
  - Image Segmentation.
  - Speech Recognition.
- Probabilistic Graphical Models are generic structured prediction models.
  - Bayesian Networks
  - Markov Random Fields
  - Sequence-to-Sequence Models (in a future lecture).
- Other models also considered (e.g. Structured SVM)

Structured Output Prediction

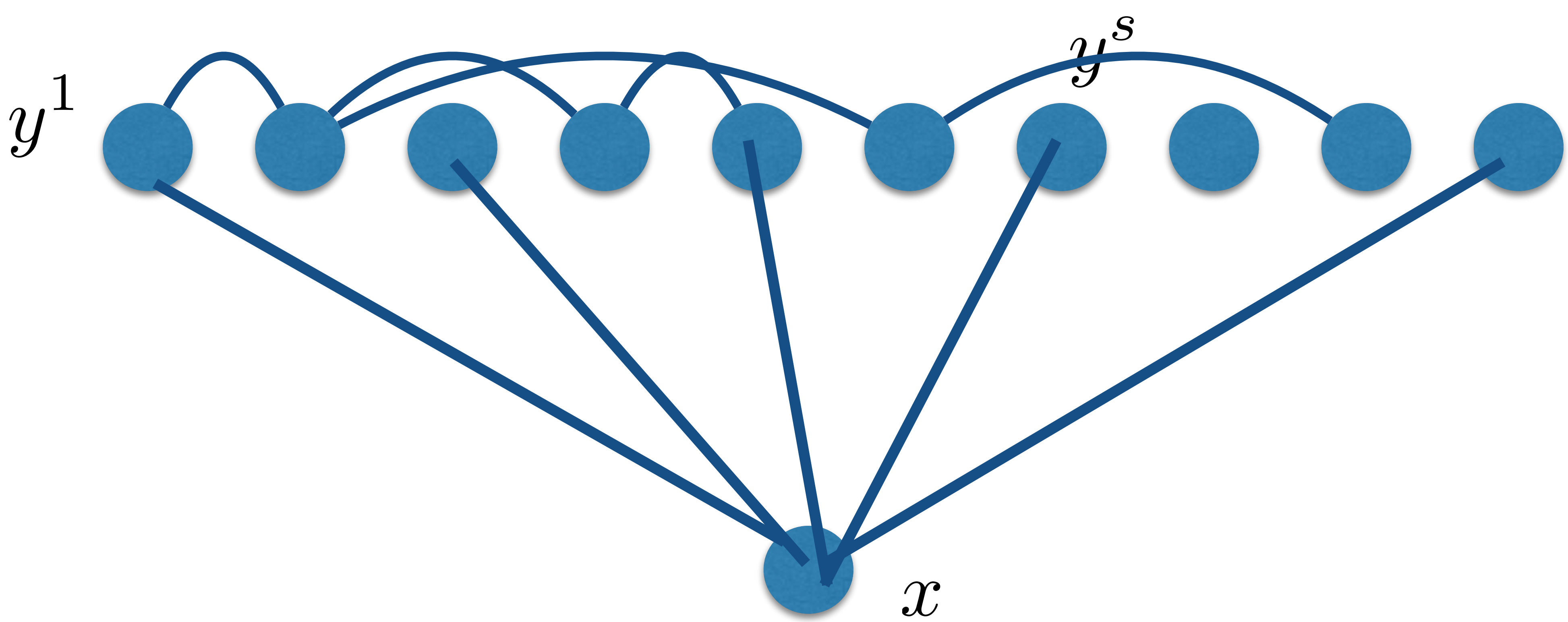
- Suppose $y = (y^1, \ldots, y^s, \ldots)$.



- If $p(y \mid x) = \prod_i p(y^i \mid x)$, the outputs are conditionally independent: we can estimate them separately.

Structured Output Prediction

- Suppose $y = (y^1, \ldots, y^s, \ldots)$.

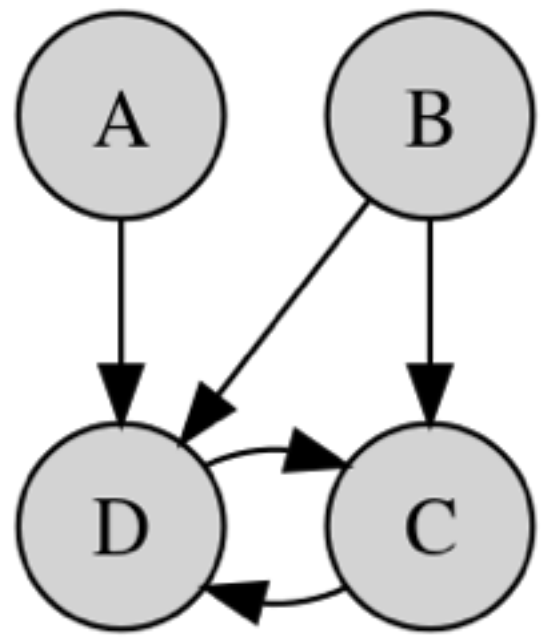


- But when we introduce statistical dependencies across outputs, the general model becomes

$$p(y \mid x) = \frac{\exp\left(-F(y, x, \Theta)\right)}{Z}$$

Graphical Models

- Broad class of probabilistic models that express a joint distribution as a product of factors.
- The dependency is expressed in terms of a graph:



(source: wikipedia)

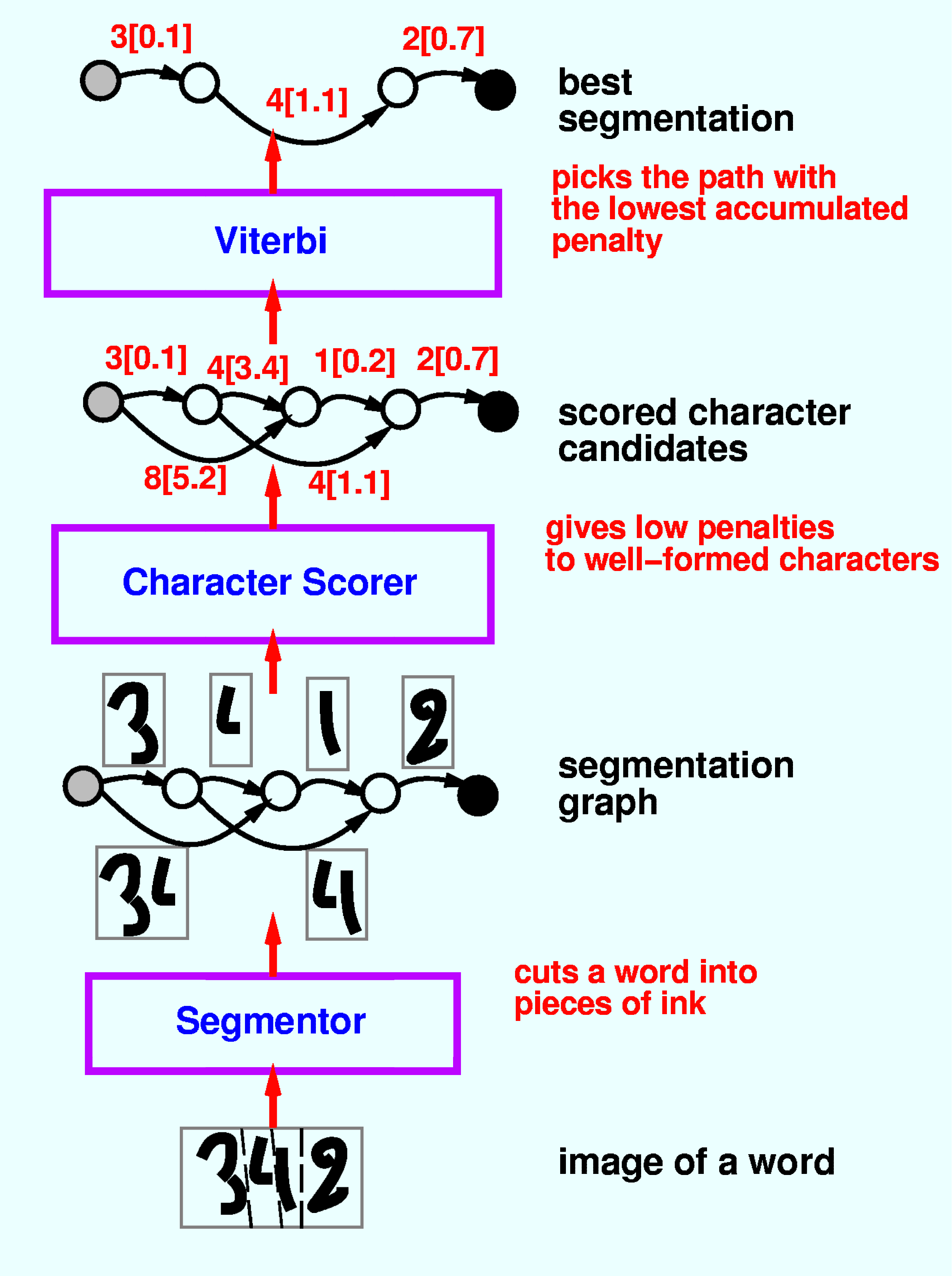
$$p(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i \mid \mathcal{P}_i)$$

$\mathcal{P}_i$: context associated with X_i

- Many instances: trees, factor graphs, Restricted Boltzmann machines (more on that later), Markov random fields,…

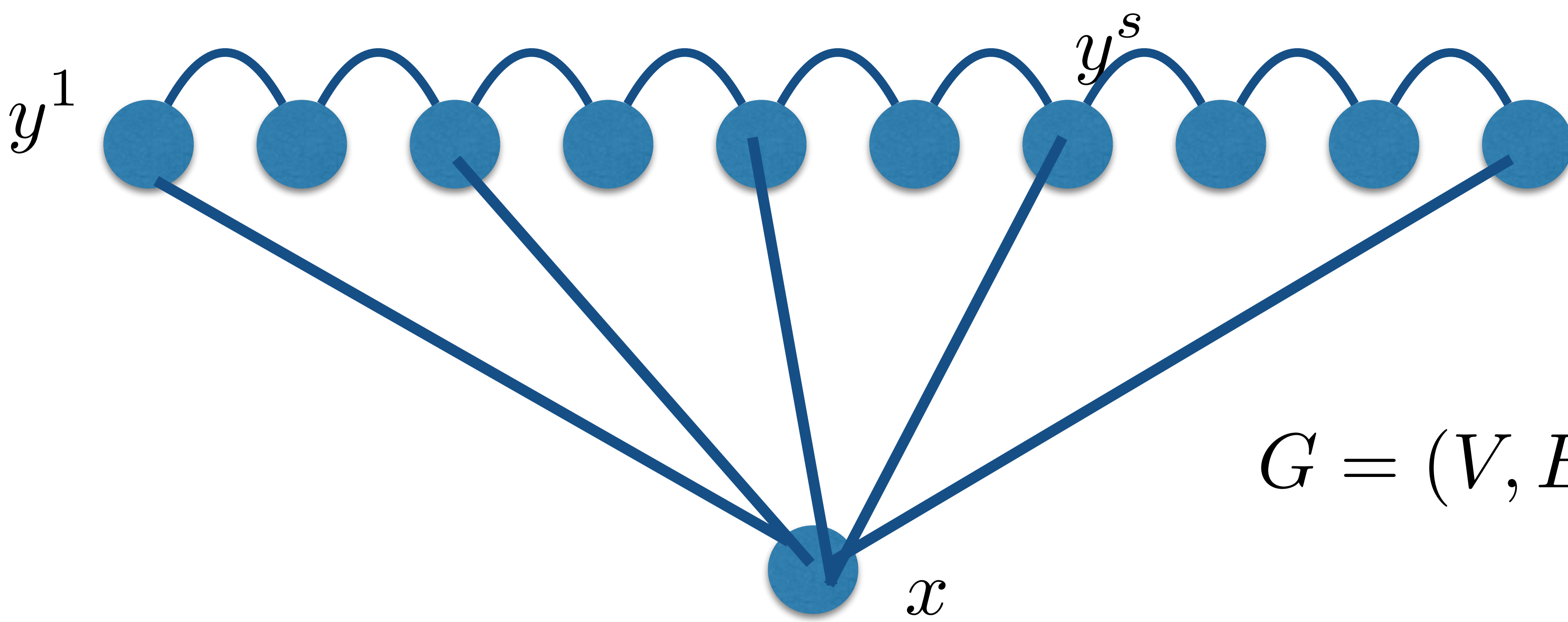
Graph Transformer Network

- [Bottou, Bengio & LeCun, '97]
- Graphical model over possible "segmentations" of handwritten characters
- Used commercially to read ~10% checks in the US (1996).



Conditional Random Fields

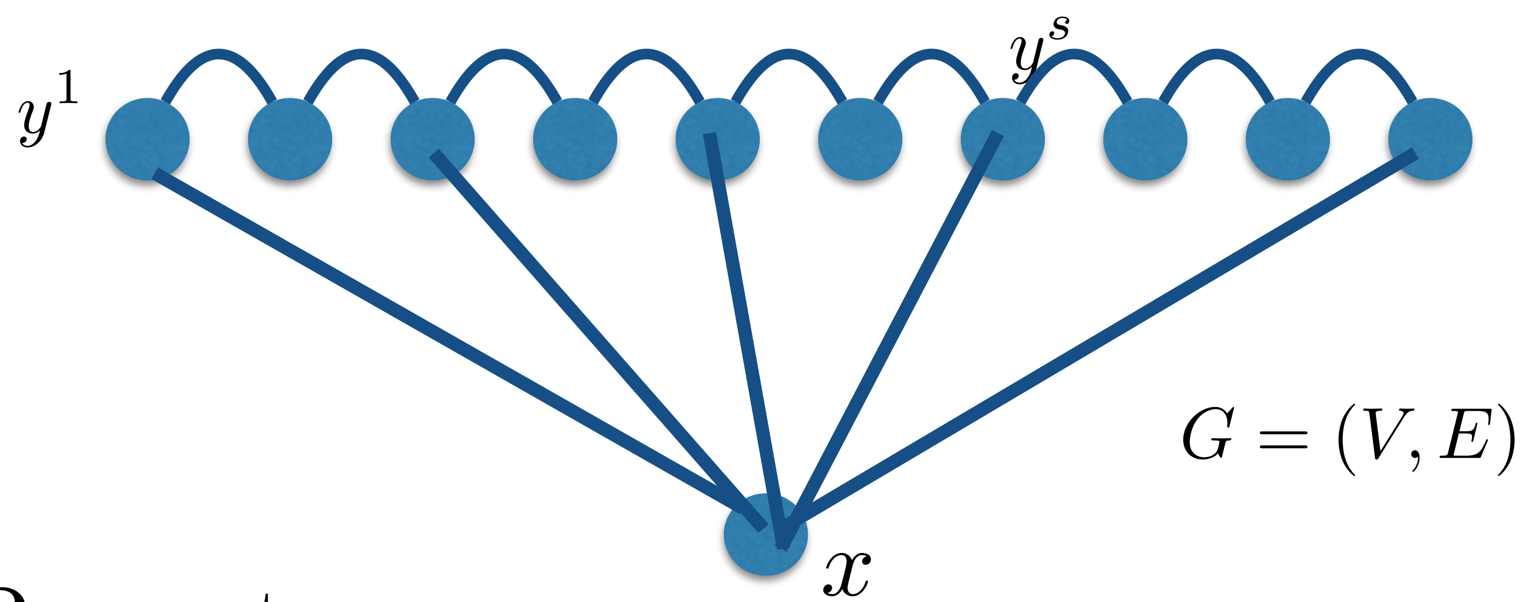
- Many problems ask to predict an output with temporal or spatial structure (eg speech, image (segmentation), natural language text).
- A Markov Random Field is a graphical model on an undirected graph:



$$G = (V, E)$$

Conditional Random Fields

- A Markov Random Field is a graphical model on an undirected graph:



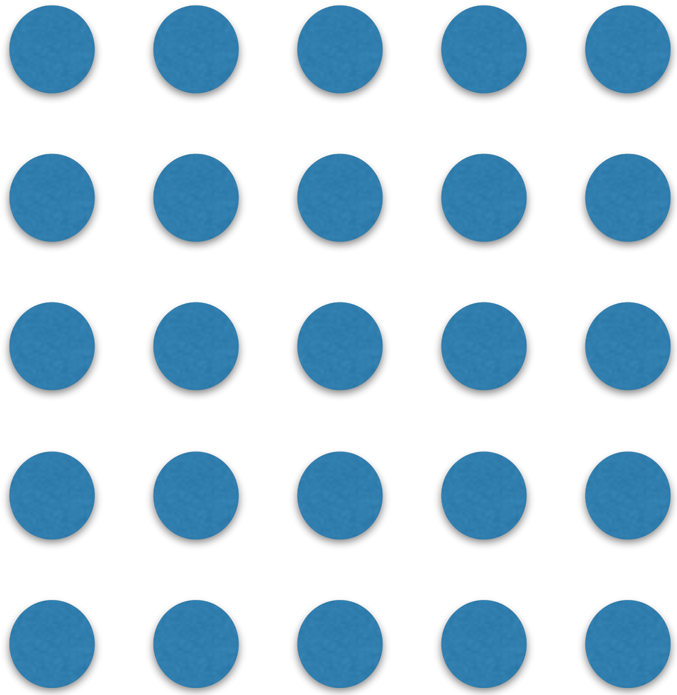
Markov Property:

$$p(y_i \mid X, y_j, j \neq i) = p(y_i \mid X, y_j, j \sim i)$$

- Inference is intractable for general graphs
  - trees and chains are exceptions
  - Algorithms for approximate inference: message passing, Viterbi, mean field inference.

Conditional Random Fields

- In images, pixels form a 2D lattice graph:



- In pixel labeling tasks (ie segmentation), the output configuration probability is expressed as

$$p(y \mid x) = \frac{e^{-E(y,x)}}{Z} \;, \;\; (Z: \;\; \text{partition function})$$

Conditional Random Fields

$$p(y \mid x) = \frac{e^{-E(y,x)}}{Z} \;, \;(Z:\; \text{partition function})$$

$$E(y,x) = \sum_{u} \psi_u(y,x) + \sum_{u \neq v} \psi_{u,v}(y,x)$$

ψ_u: "unary" potentials measure cost of pixel u being labeled y_u.

$\psi_{u,v}$: pairwise potentials measure cost of jointly assigning labels y_u, y_v at pixels u and v.

- unary potentials predict labels at each location as if they were independent from the rest
- pairwise potentials provide data-dependent smoothing.

CRFs as Convolutional Neural Networks

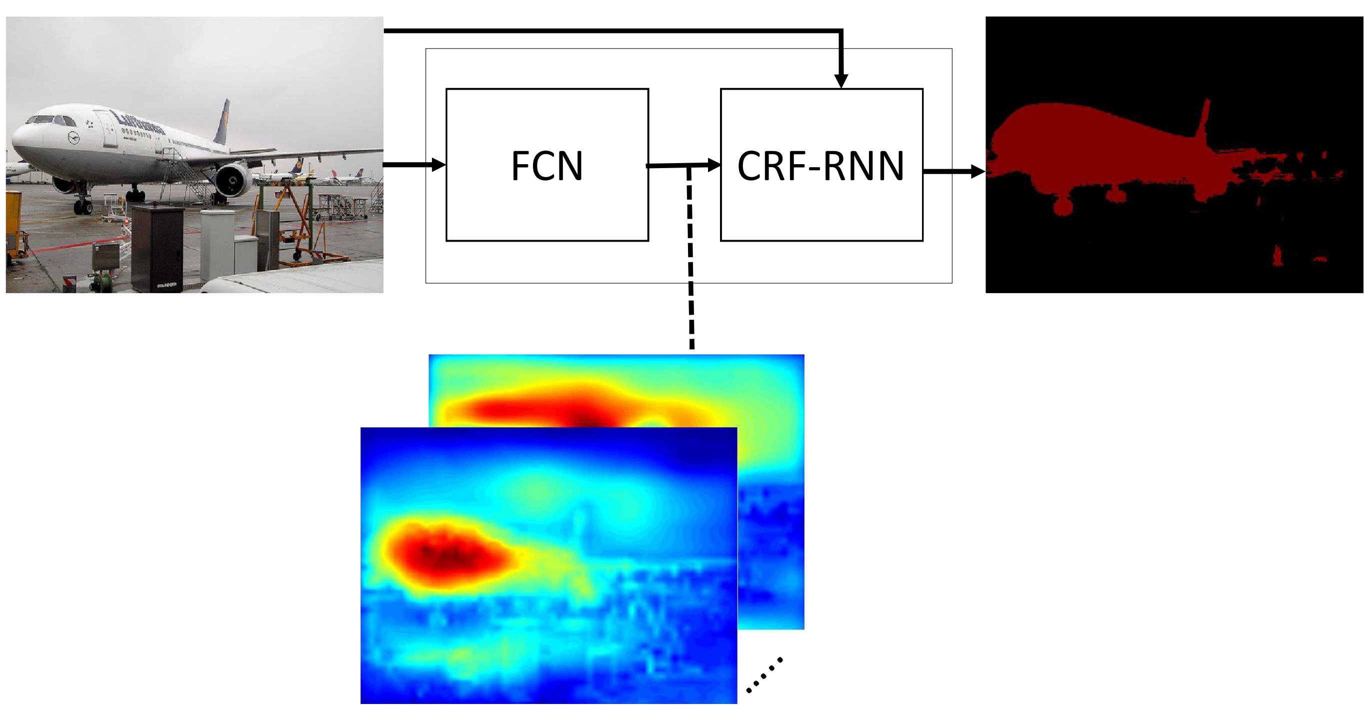
- An approximate posterior inference for the CRF model is done with mean-field approximation:

$$\text{Approximate } p(y \mid x) \text{ with } q(y \mid x) = \prod_i q_i(y_i \mid x) \text{ iteratively.}$$

- One can also consider belief propagation as an alternative to mean-field approximation (see http://www.eecs.berkeley.edu/~wainwrig/Talks/A_GraphModel_Tutorial for a great tutorial!)

CRFs as Convolutional Neural Networks

- [Zheng et al,'15] approximate the mean-field message passing iterations with CNN layers with shared parameters.
- The system can be efficiently trained end-to-end.



Algorithm 1 Mean-field in dense CRFs [27], broken down to common CNN operations.

$Q_i(l) \leftarrow \frac{1}{Z_i} \exp\left(U_i(l)\right)$ for all i ▷ Initialization

while not converged **do**

$\tilde{Q}_i^{(m)}(l) \leftarrow \sum_{j \neq i} k^{(m)}(\mathbf{f}_i, \mathbf{f}_j) Q_j(l)$ for all m ▷ Message Passing

$\check{Q}_i(l) \leftarrow \sum_m w^{(m)} \tilde{Q}_i^{(m)}(l)$ ▷ Weighting Filter Outputs

$\hat{Q}_i(l) \leftarrow \sum_{l' \in \mathcal{L}} \mu(l, l') \check{Q}_i(l)$ ▷ Compatibility Transform

$\breve{Q}_i(l) \leftarrow U_i(l) - \hat{Q}_i(l)$ ▷ Adding Unary Potentials

$Q_i \leftarrow \frac{1}{Z_i} \exp\left(\breve{Q}_i(l)\right)$ ▷ Normalizing

end while

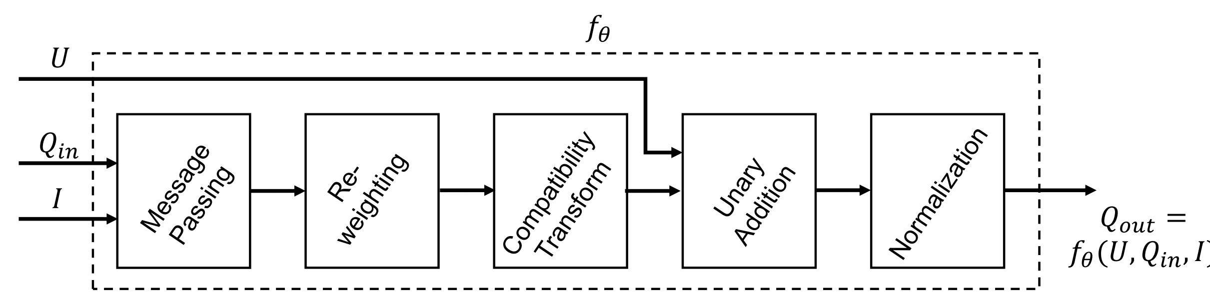
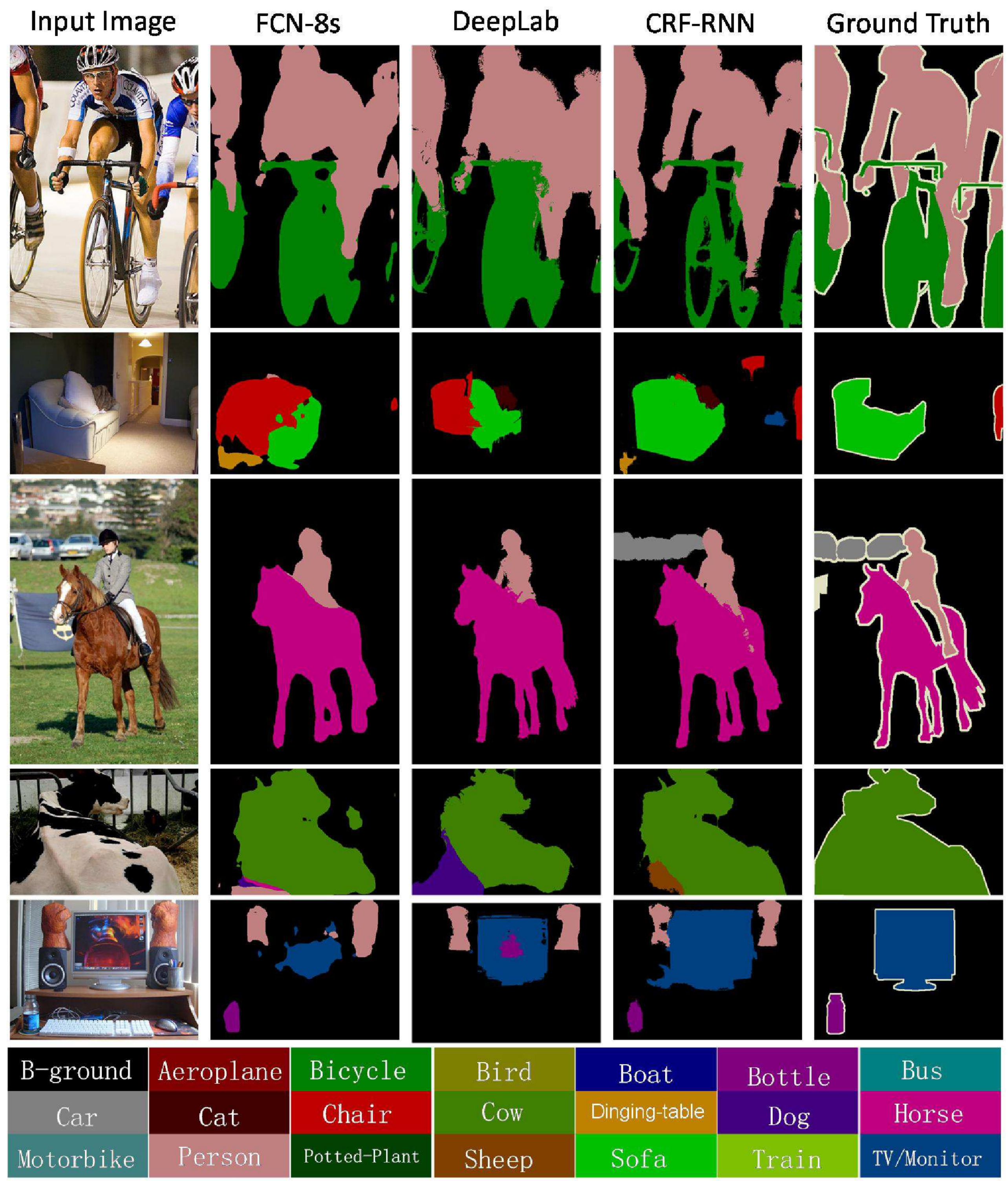


Figure 1. **A mean-field iteration as a CNN.** A single iteration of the mean-field algorithm can be modelled as a stack of common CNN layers.

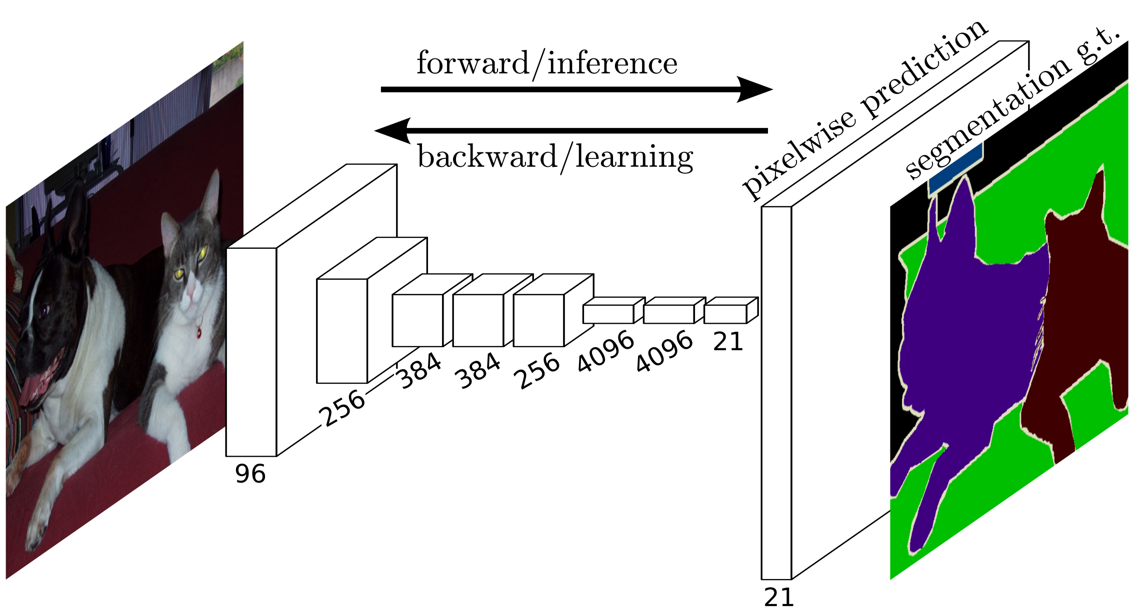
Example: Segmentation

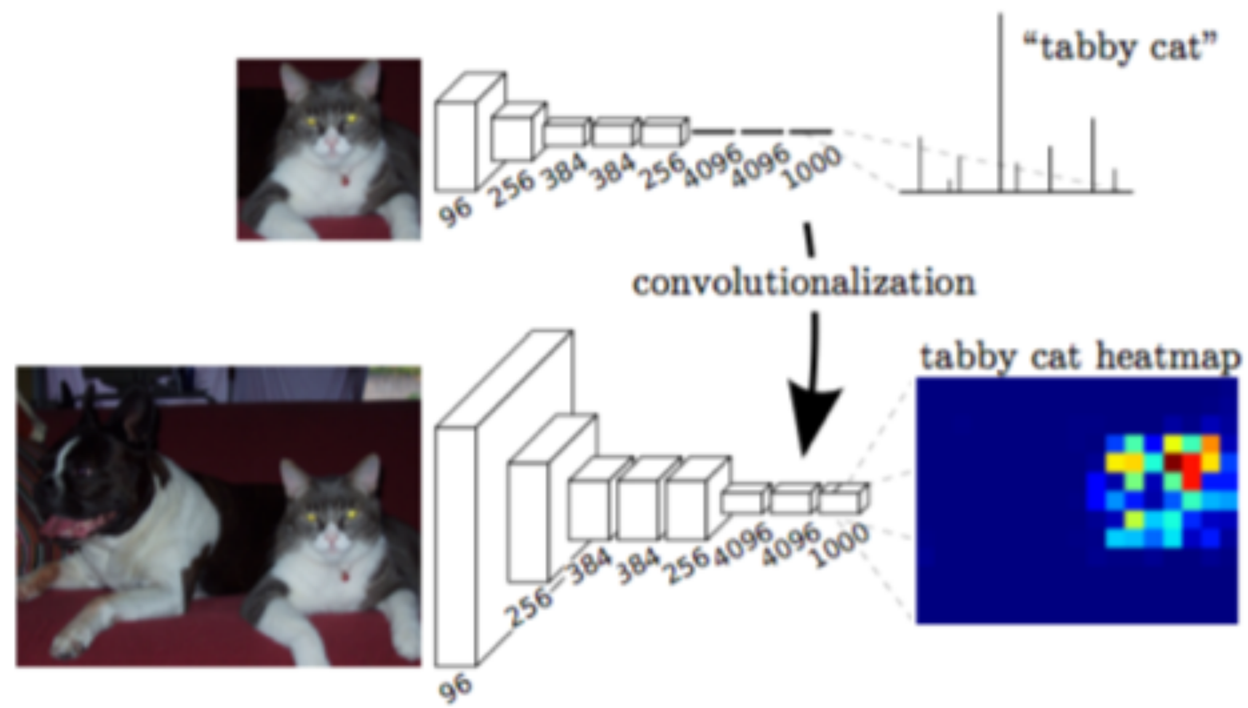
- Results from [Zheng et al, ICCV'15]



Example: Segmentation

- The CRF approximation is a specific CNN model.
- [Long, Shelhamer et al,CVPR'15] proposed a simpler CNN architecture that also produces excellent results.
- Idea: Combine outputs from different layers and refine the spatial resolution of the output.





- See also "Learning to Segment Object Candidates" [Pinheiro et al'15].

Example: Human Pose Estimation

- Human muscle joints are *very structured.*
- [Tompson et al, NIPS'14] considered a joint training of CNN and Markov Random Fields.

Example: Pose Estimation

- The unary potentials are modeled as detection CNNs.
- The pairwise potentials between different parts are modeled as convolutional priors.
- The marginal likelihoods for each part are of the form

$$\bar{p}(A \mid x) = \frac{1}{Z} \prod_{B} (p(A \mid B, x) \star p(B \mid x) + b_{B \to A})$$

($b_{B \to A}$: bias term for the mssage from B to A)