Stat 212b: Topics in Deep Learning
Lecture 2

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Objectives

- Classification, Kernels and metrics
- Representations for recognition
  - curse of dimensionality
  - invariance/covariance
  - discriminability
- Variability models
  - transformation groups and symmetries
  - deformations
  - stationarity
  - clutter and class-specific
- Examples

High-dimensional Recognition Setup

- Input data x lives in a high-dimensional space:

$$x \in \Omega,\ \ \Omega \subset \mathbb{R}^d \qquad \text{finite-dimensional (but large } d\text{!)}$$

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- We observe (x_i, y_i) , $i = 1 \ldots n$, where

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$$y_i \in \mathbb{R} \qquad \text{(regression)}$$

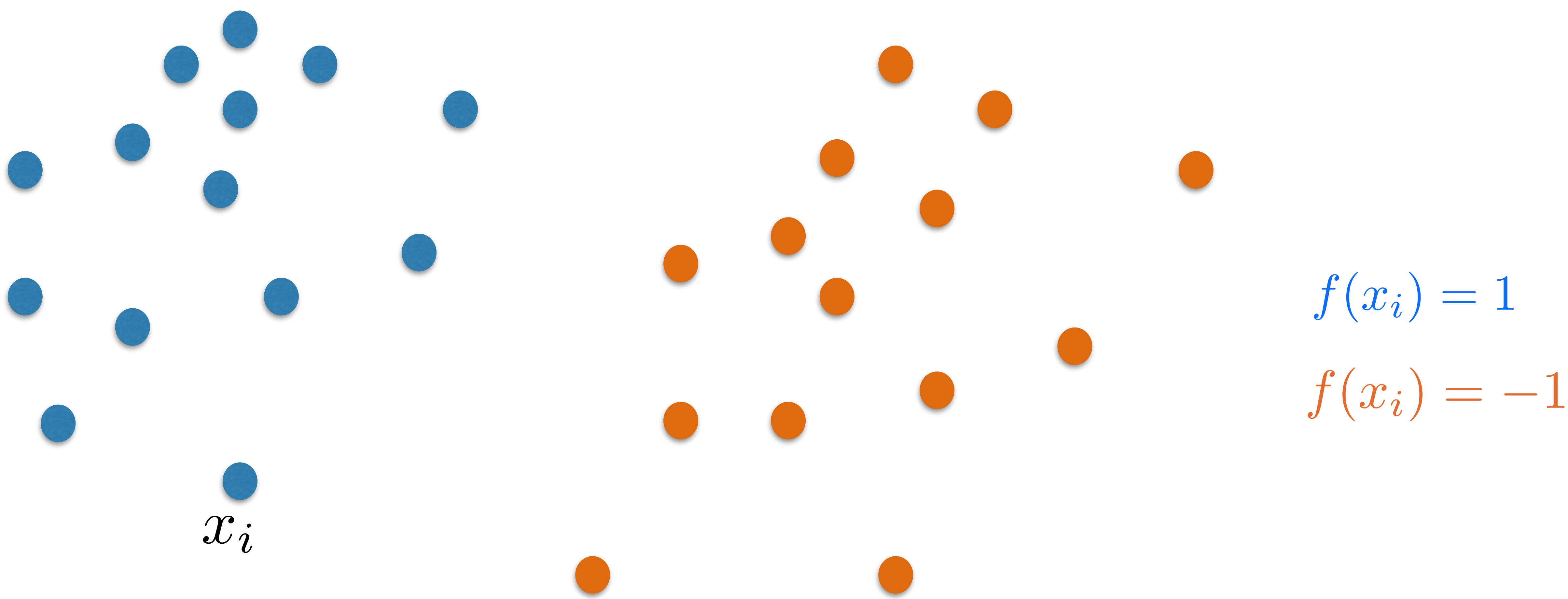
$$y_i \in \{1, K\}\ . \quad \text{(classification)}$$

- We can reduce the former to "interpolating" a function

$$f\ :\ \Omega \to \mathbb{R}^K \qquad (f(x) = p(y \mid x) \text{ in the classification case})$$

How to "interpolate" in high-dimensions?

- Let's start with a (very) simple low-dimensional setting:



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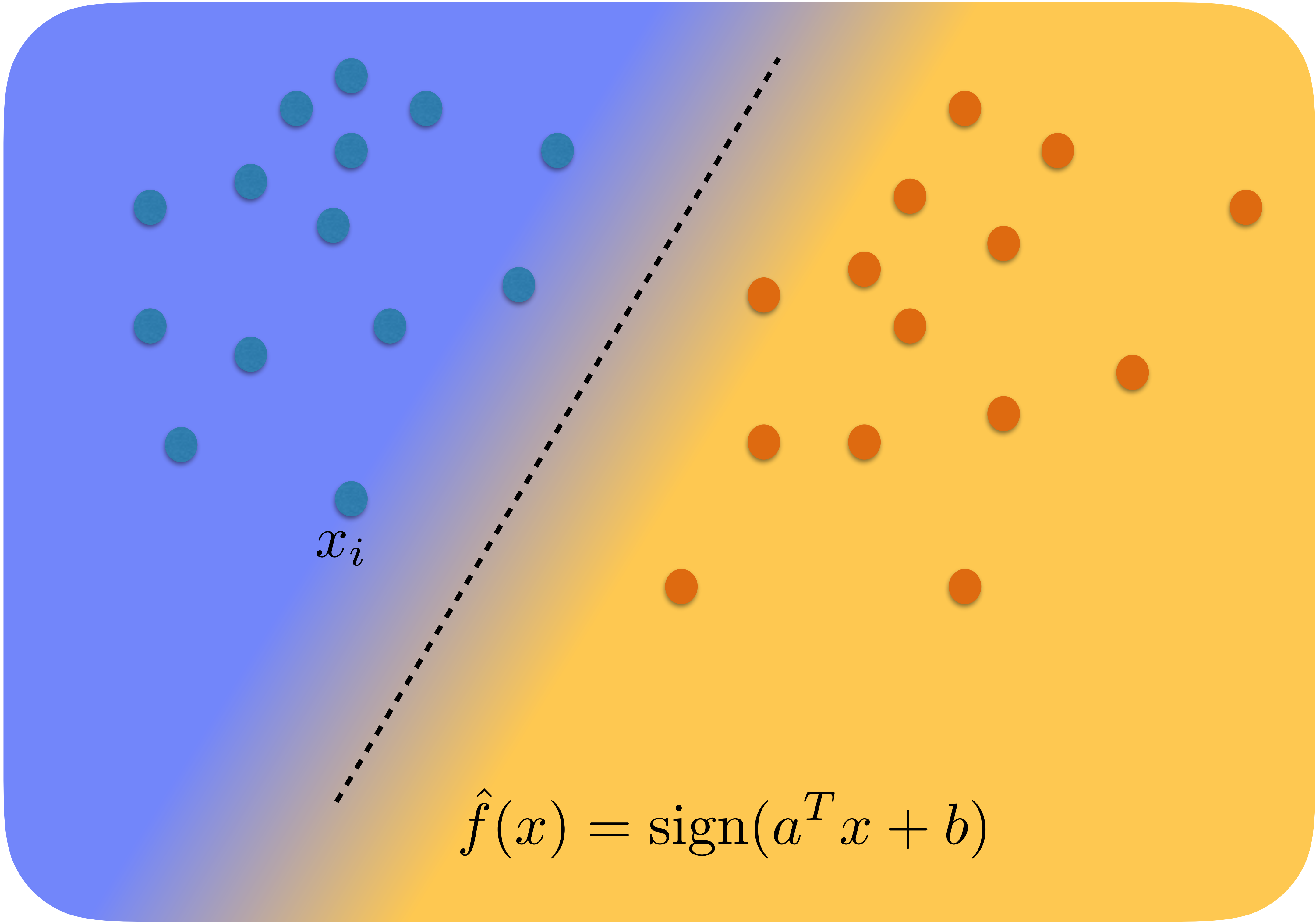
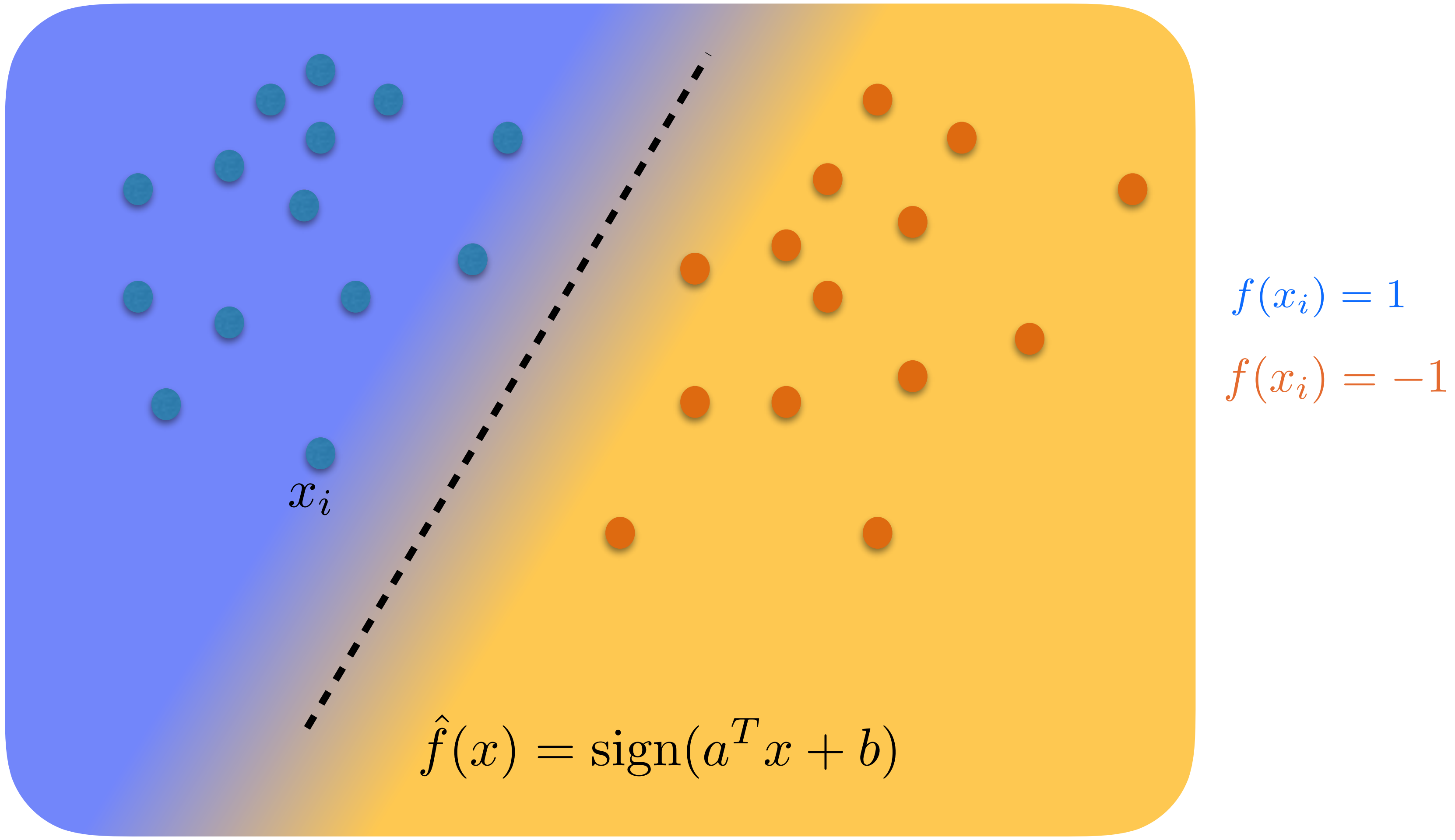


$f(x_i) = 1$

$f(x_i) = -1$

How to interpolate in high dimensions?



- We have found (linear) features $\Phi(x) = a^T x$ such that

$$|f(x) - f(x')| \leq C \|\Phi(x) - \Phi(x')\|$$

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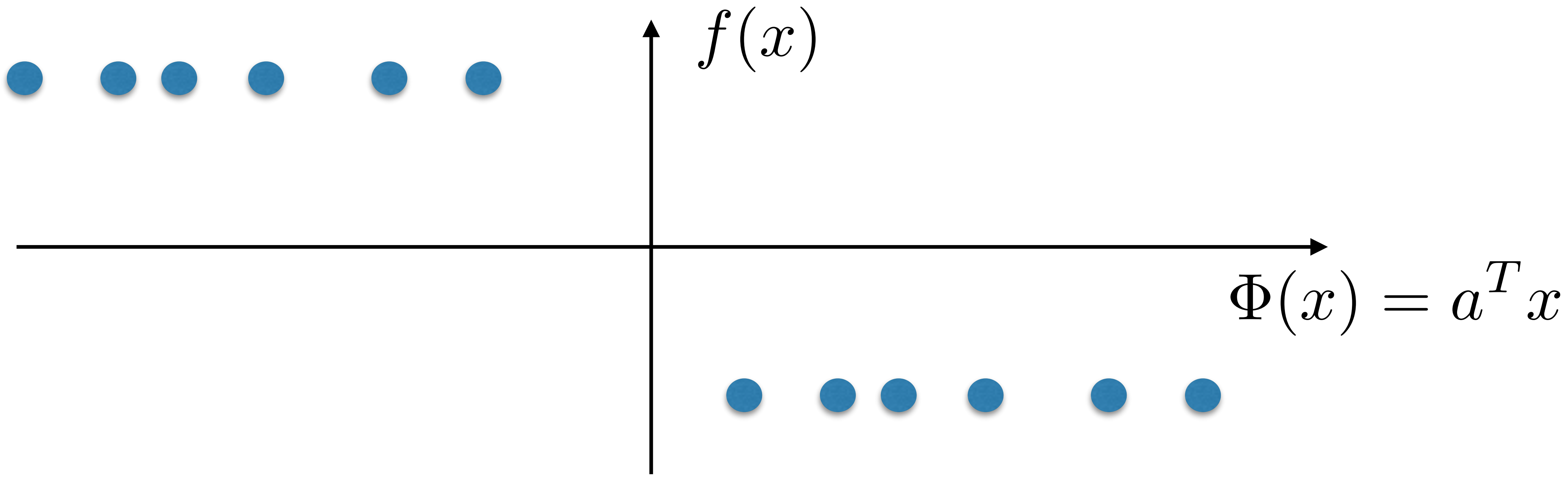
- By projecting $\Phi(x) = a^T x$ we transform the high-dimensional problem into a simple low-dimensional interpolation problem:



Support Vector Machines

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- Empirical Risk Minimization:

$$\min_{a,b} \frac{1}{n}\sum_{i=1}^{n} \ell(y_i, \hat{f}(x_i)) + \lambda \|a\|^2 ,$$

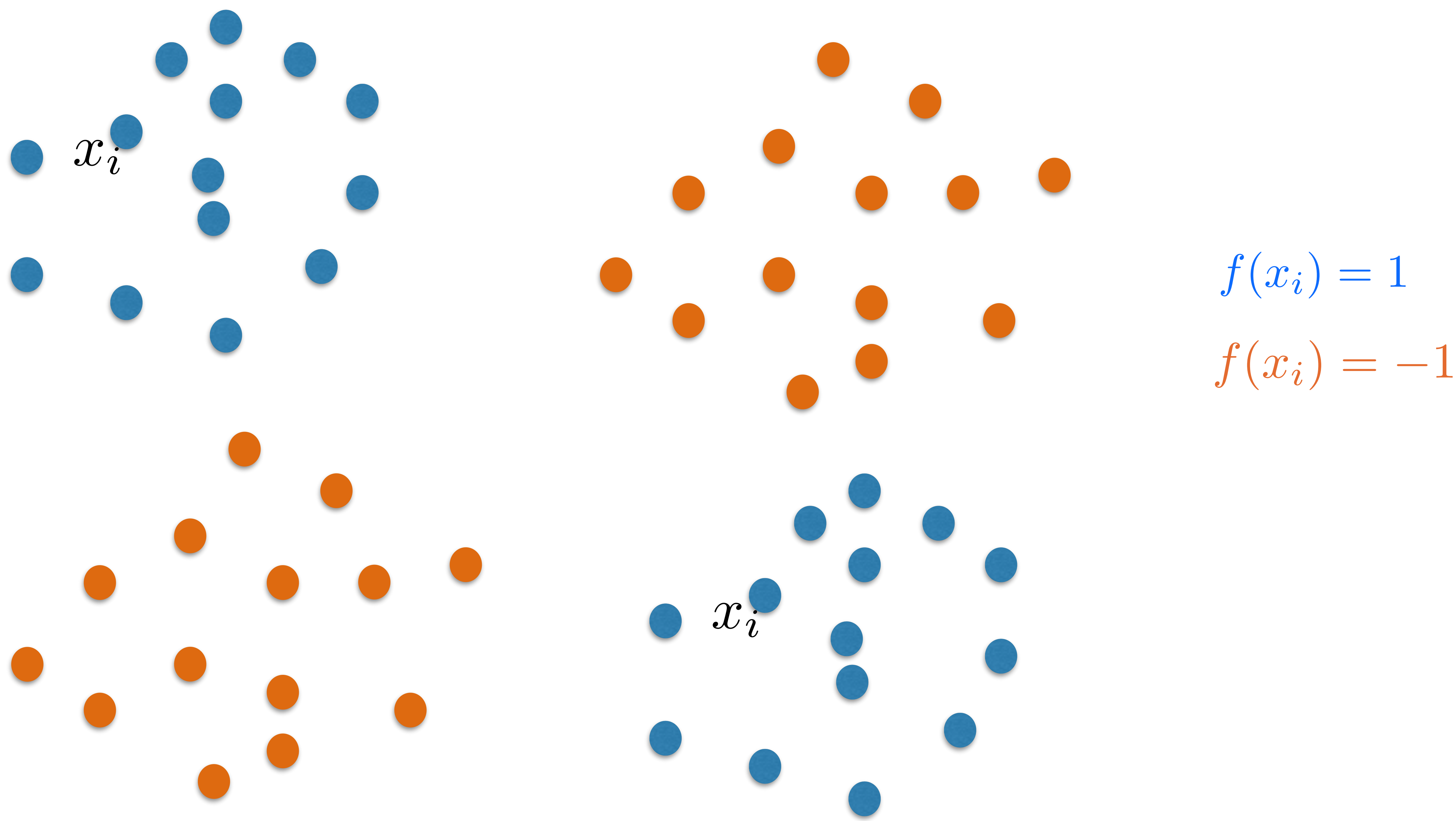
enforces training examples
to fall in the right side of the hyperplane

enforces large margin

$$\ell(y, \hat{y}) = \max(0, 1 - y \cdot \hat{y}) \quad : \text{hinge loss} .$$

SVMs and Kernels

- Not all problems are linearly separable:



the "XOR" problem

- By using the Lagrangian dual of the previous program, we can rewrite our previous solution as

$$\hat{f}(x) = \text{sign}\left(\sum_i \alpha_i y_i K(x_i, x)\right) \ ,$$

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- We can replace the linear kernel by a non-linear one, eg
  - polynomial: $K(x, y) = \langle x, y \rangle^d$.
  - Gaussian radial basis function: $K(x, y) = \exp(-\|x - y\|^2 / \sigma^2)$.

The Kernel "trick"

- For a wide class of psd kernels (Mercer Kernels), we have a representation in terms of an inner product:

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- Is this enough to characterize good features/kernels?

Generalization Error

- It is easy to construct discriminative features:
  - Using a Gaussian RBF, it suffices to let $\sigma^2 \to 0$.
  - The estimator converges to the *nearest neighbor* classifier:

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- Underlying question: how to compare signals in high-dim?

Curse of Dimensionality

- In a finite-dimensional, bounded space, all metrics are *equivalent*:

for each $x \in \Omega$, exists constants c, C such that
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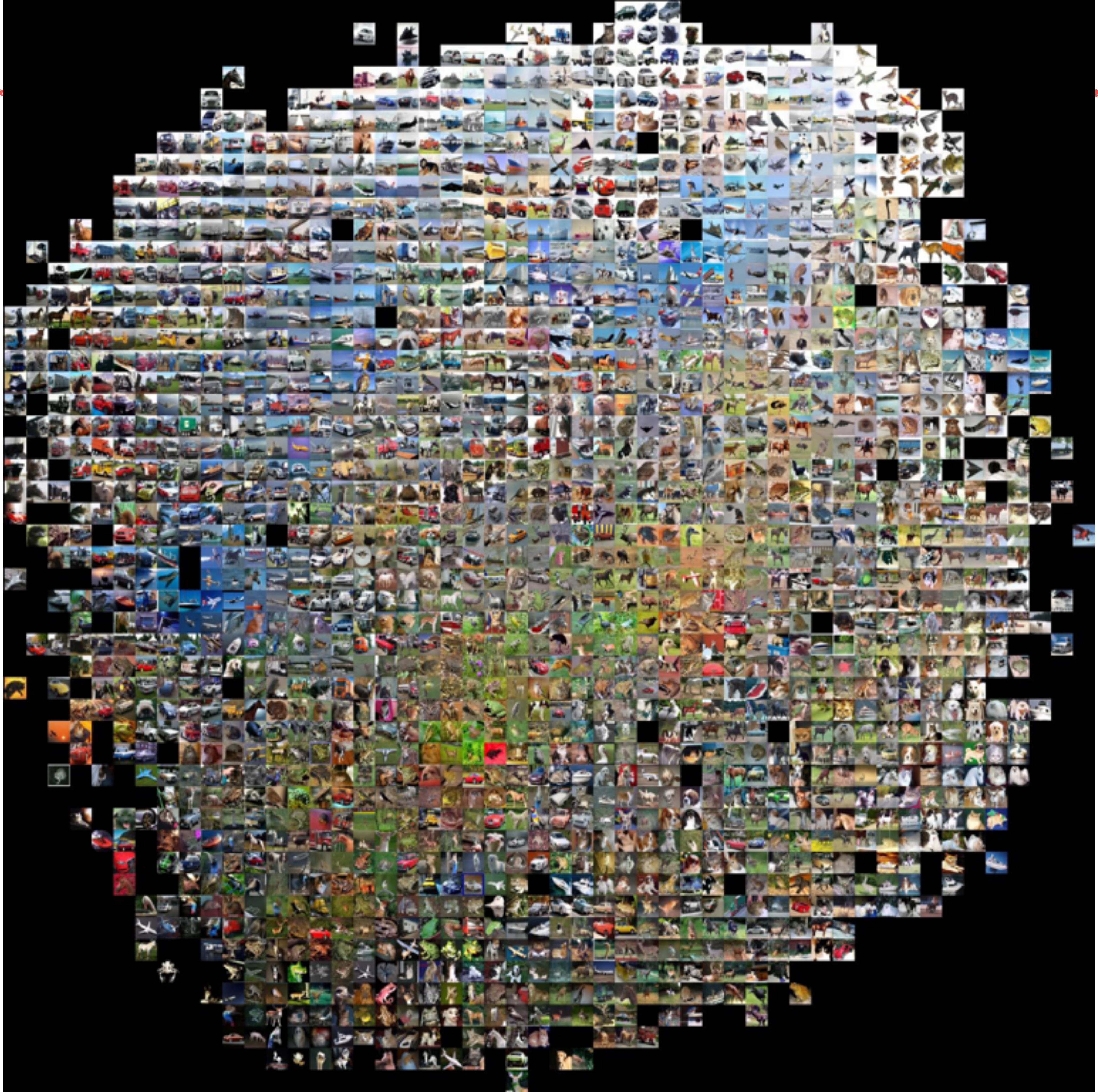
- Local decisions around training do not extend to the whole space.
- So, we need a guiding principle that plays well with our data (images, sounds, etc.)



2-dimensional embedding of CIFAR-10 using Euclidean similarity

from A. Karpathy

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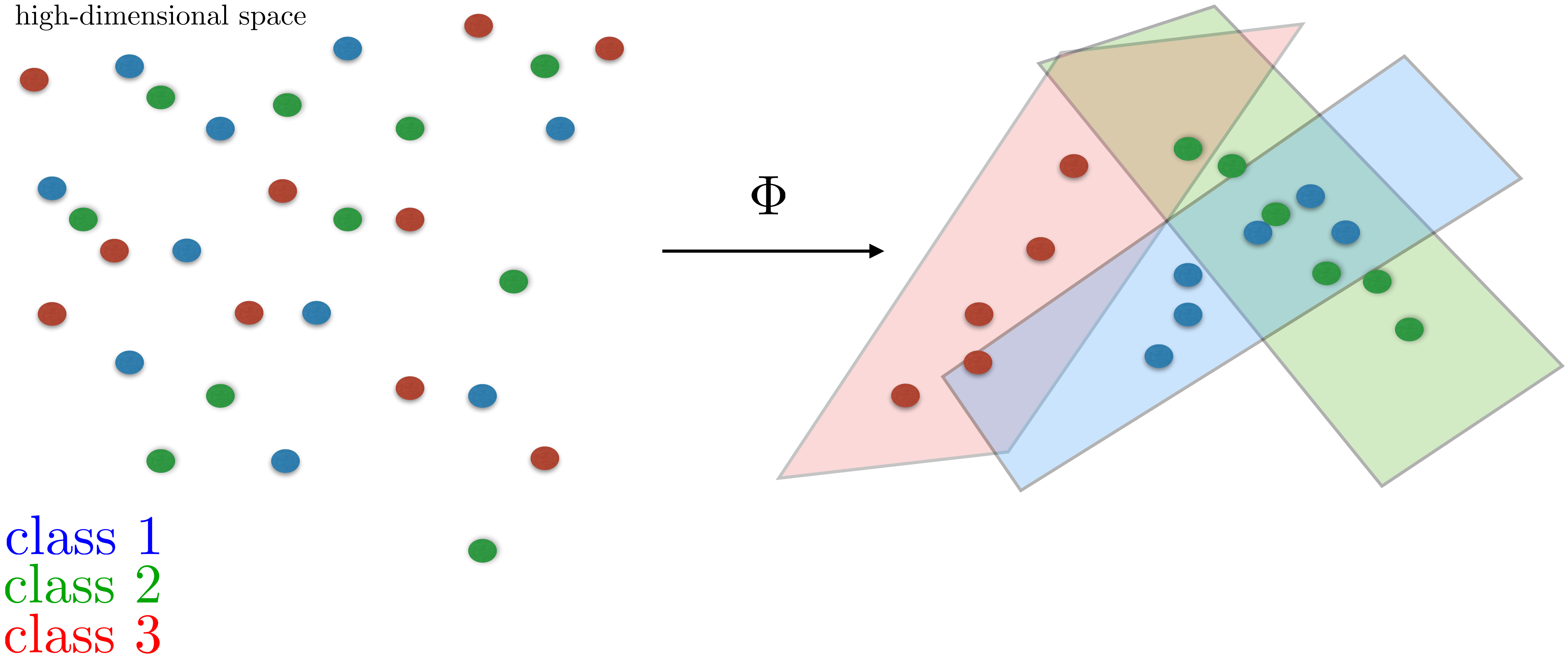
Thus the level sets of f should be mapped to parallel hyperplanes by Φ

Linearization



*In order to beat the curse of dimensionality, we need features that **linearize intra-class variability** and **preserve inter-class variability.***

Invariance and Symmetry

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Invariants: $\Phi(\varphi(x)) = \Phi(x)$ for each x.

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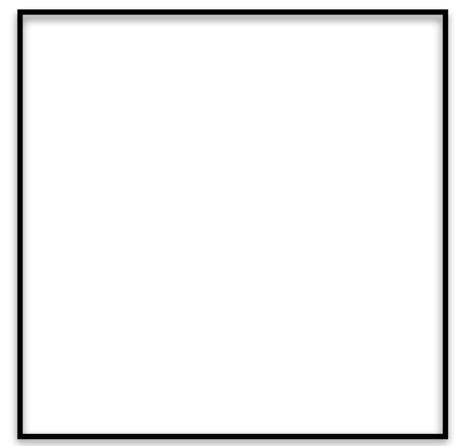
- What are those symmetries? How to impose them on Φ without breaking discriminability?

Discrete symmetries

- Which transformations leave this square unchanged?



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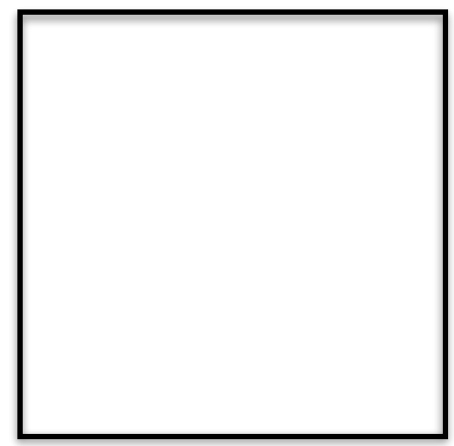

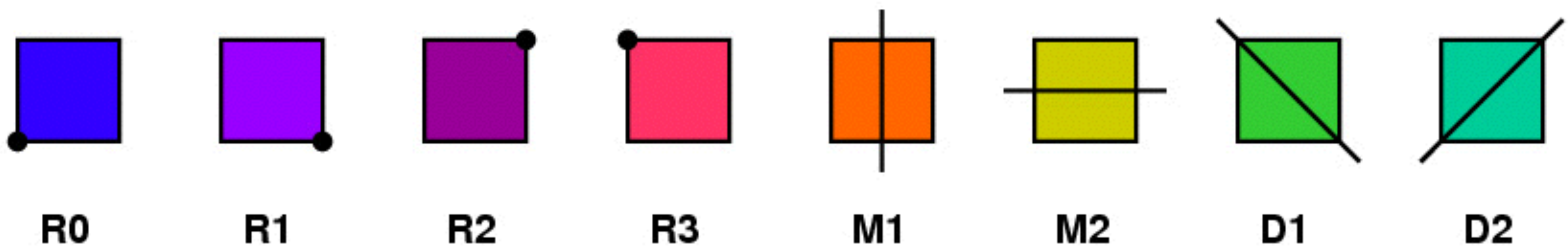

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(from http://www.cs.umb.edu/~ebl/)

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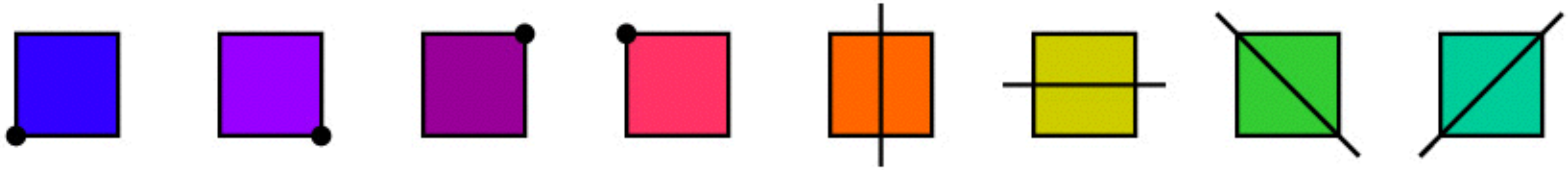

- The set of all symmetries forms a *group* G :
  - group operation: $\forall\ g_1, g_2 \in G, \quad g_1 \cdot g_2 \in G$.
  - identity element: $\exists e \in G\ s.t.\ g \cdot e = e \cdot g = g \quad \forall g \in G$.
  - inverse: $\forall\ g \in G\ \exists\, g^{-1} \in G\ s.t.\ g \cdot g^{-1} = e$.

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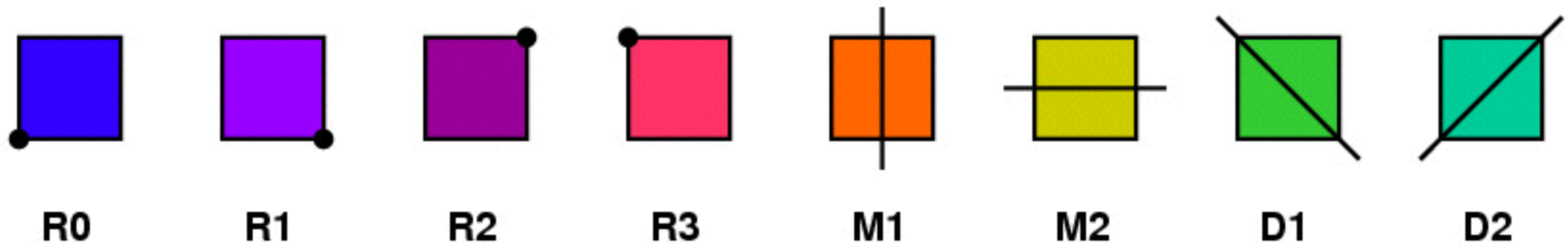
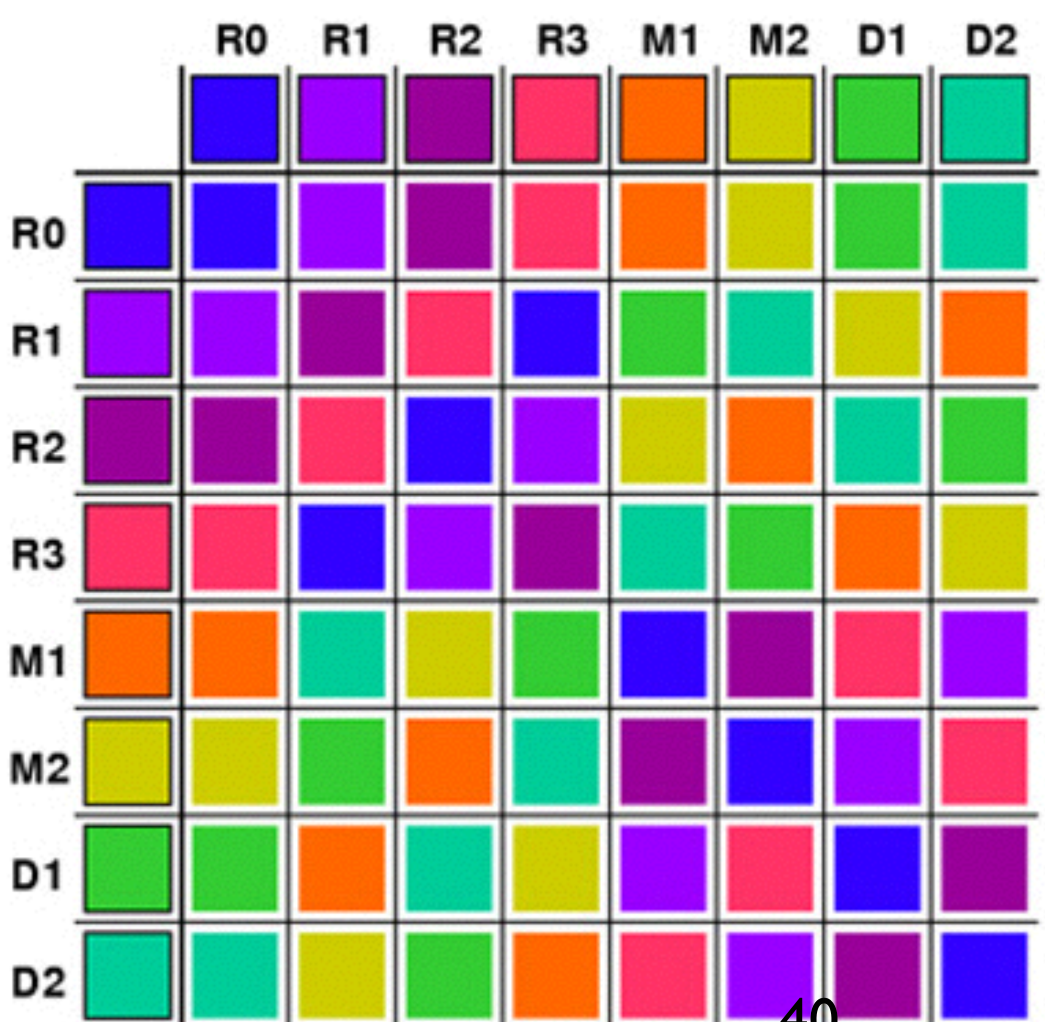


- Discrete groups are completely characterized by their multiplication table:



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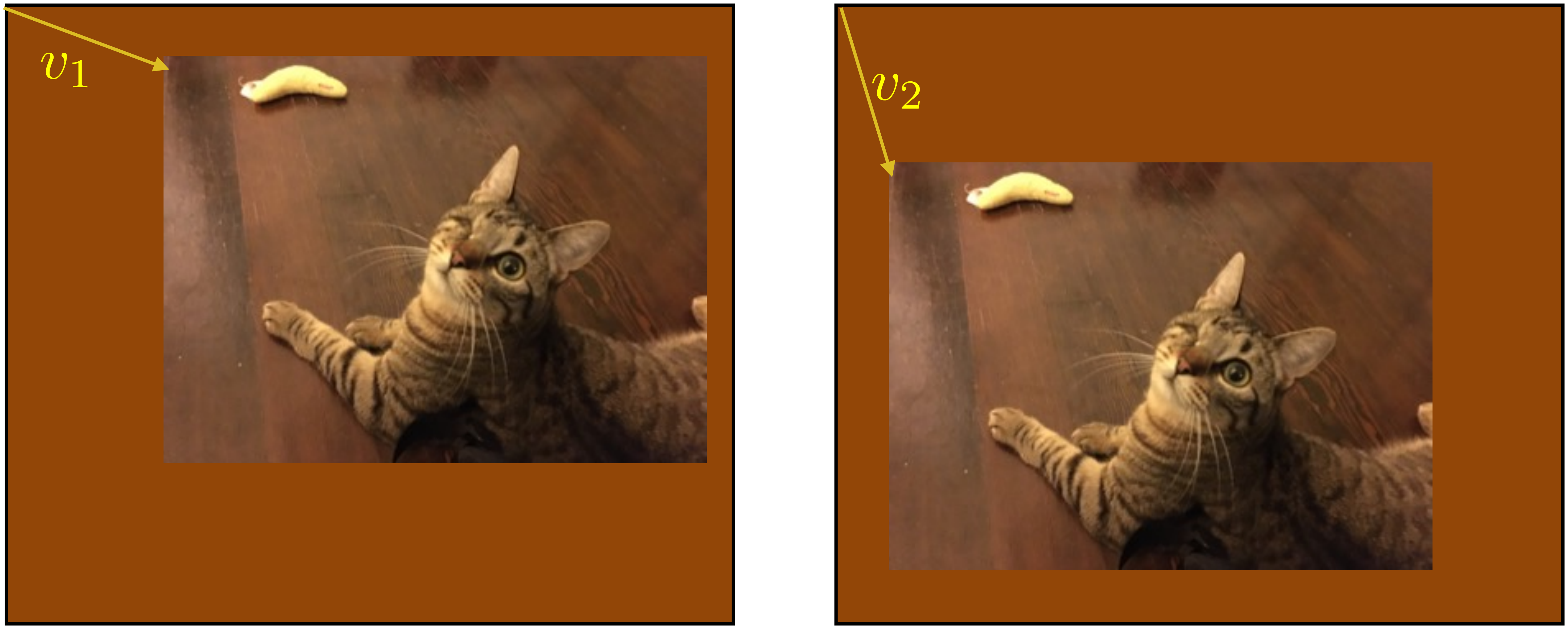

$$\text{Translations: } \{\varphi_v\,;\, v \in \mathbb{R}^2\}, \text{ with } \varphi_v(x)(u) = x(u-v).$$

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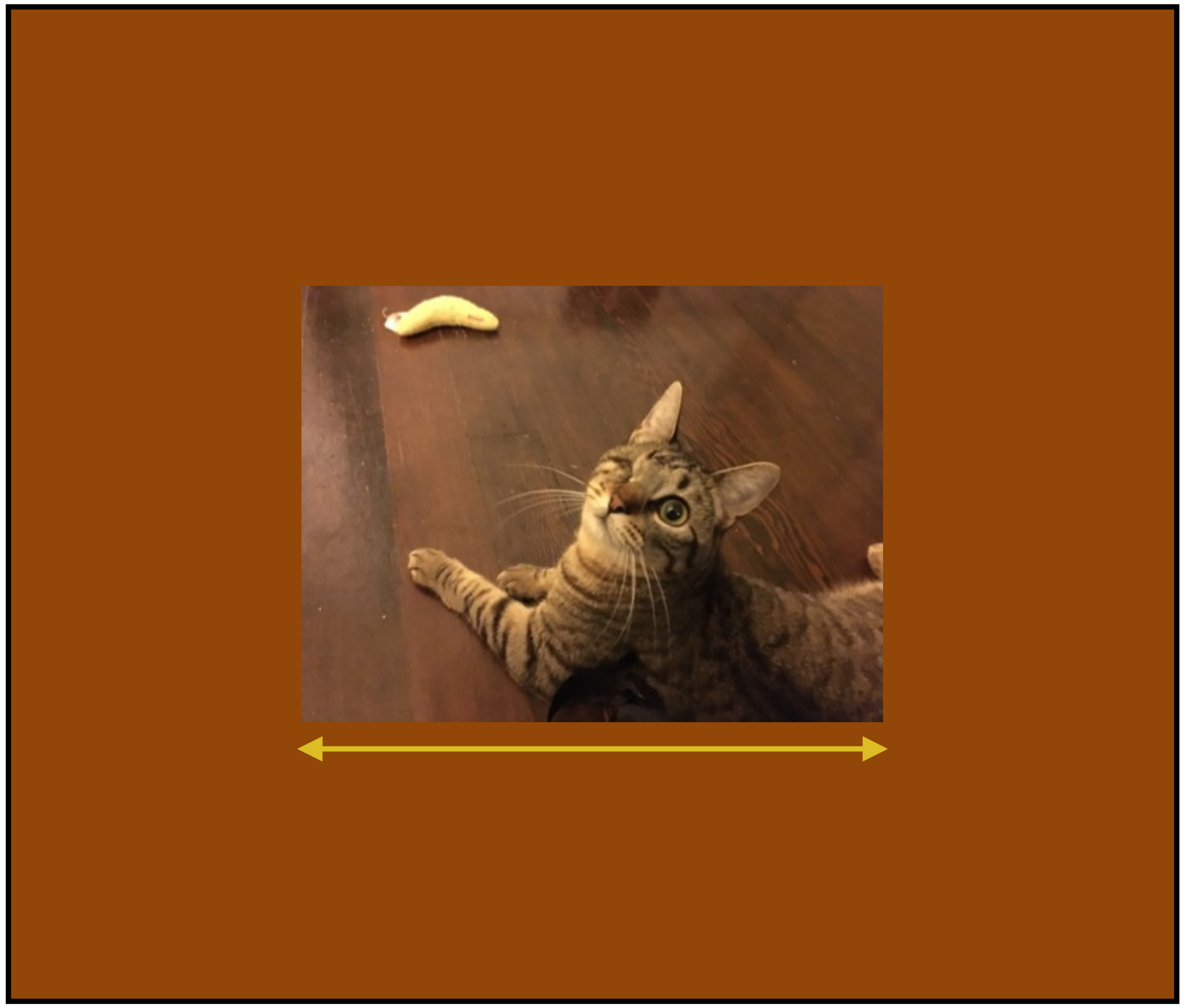

Dilations: $\{\varphi_s\,;\, s \in \mathbb{R}_+\}$, with $\varphi_s(x)(u) = s^{-1}x(s^{-1}u)$.

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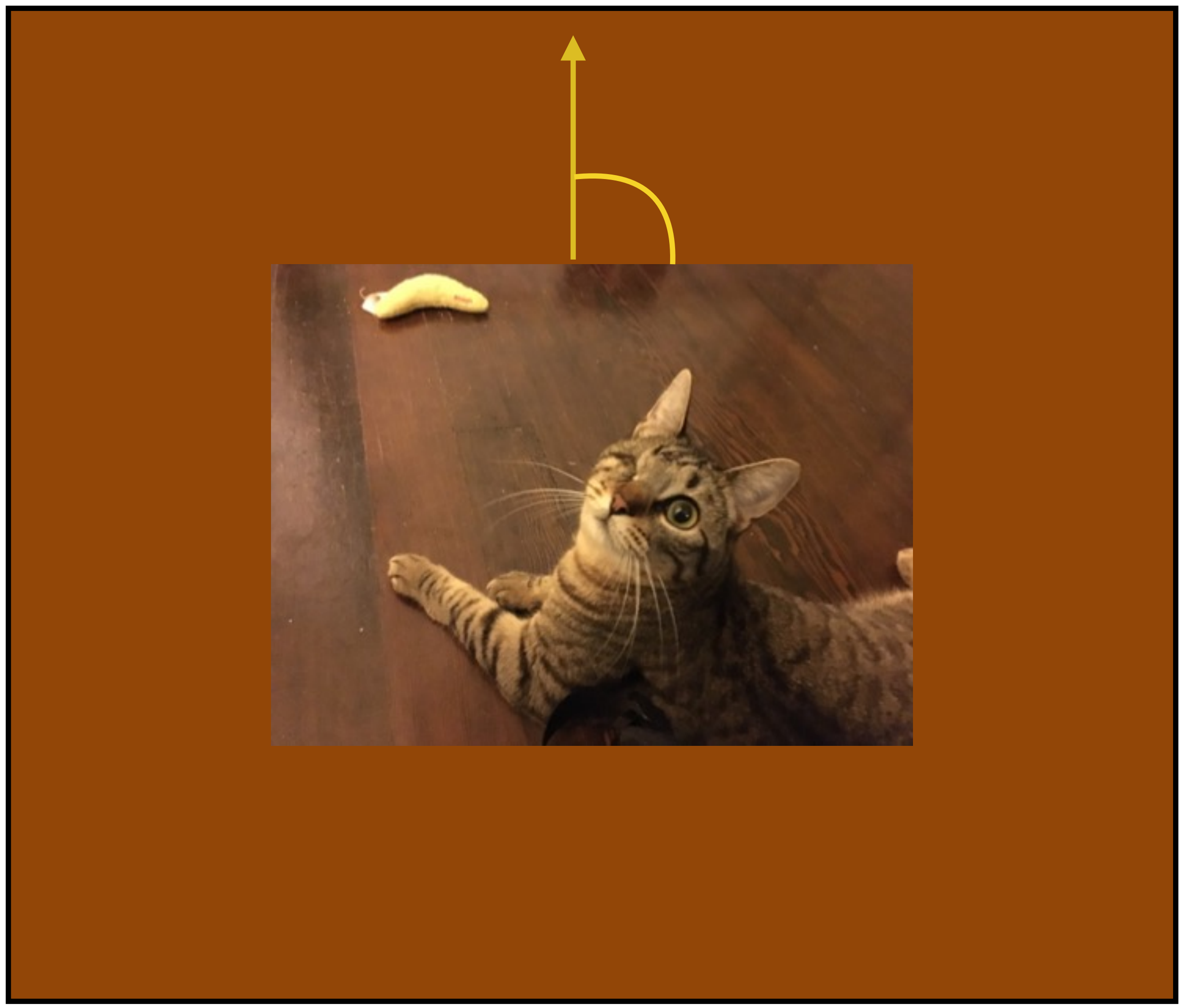

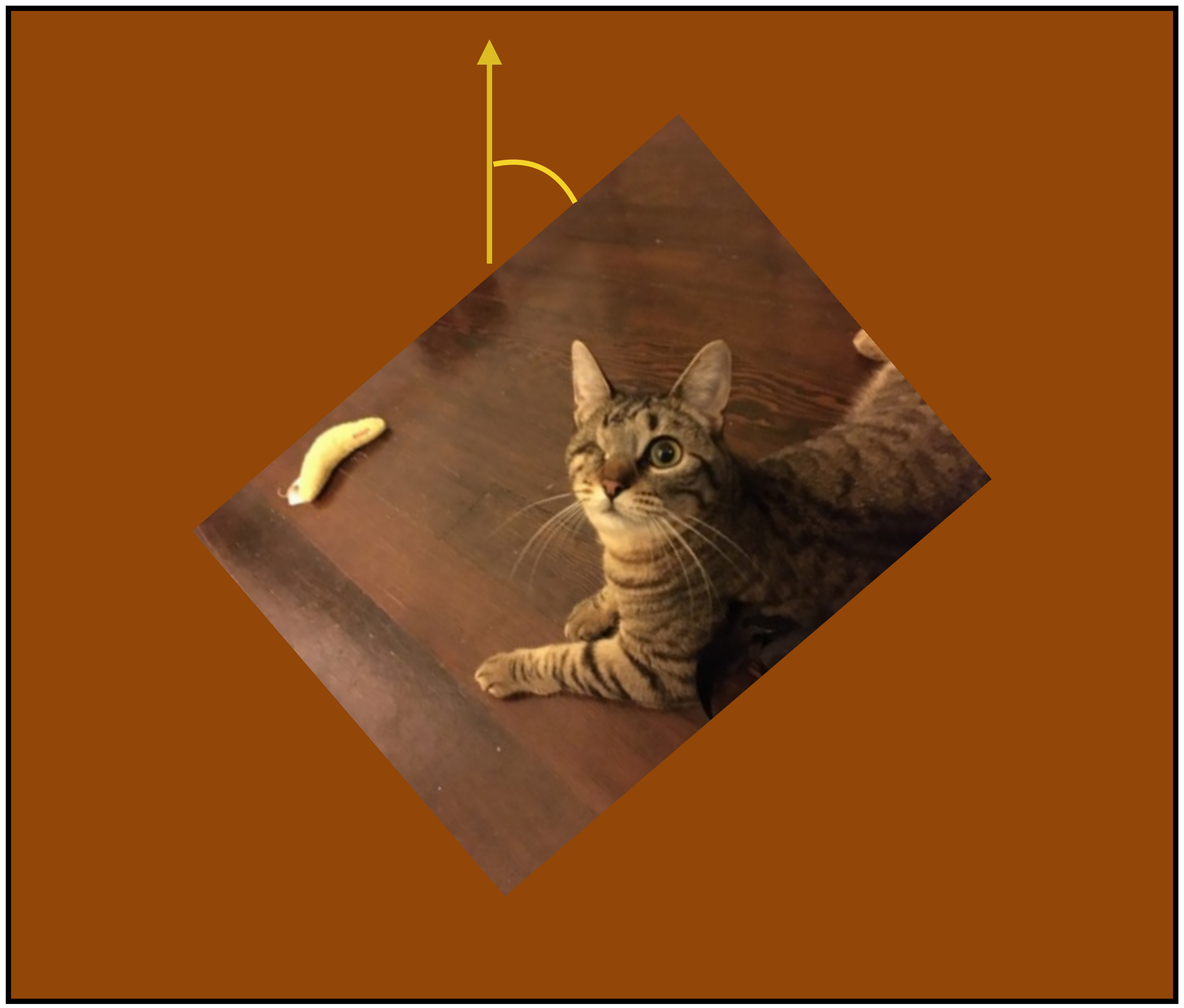

Rotations: $\{\varphi_\theta \, ; \theta \in [0, 2\pi)\}$, with $\varphi_\theta(x)(u) = x(R_\theta u)$.

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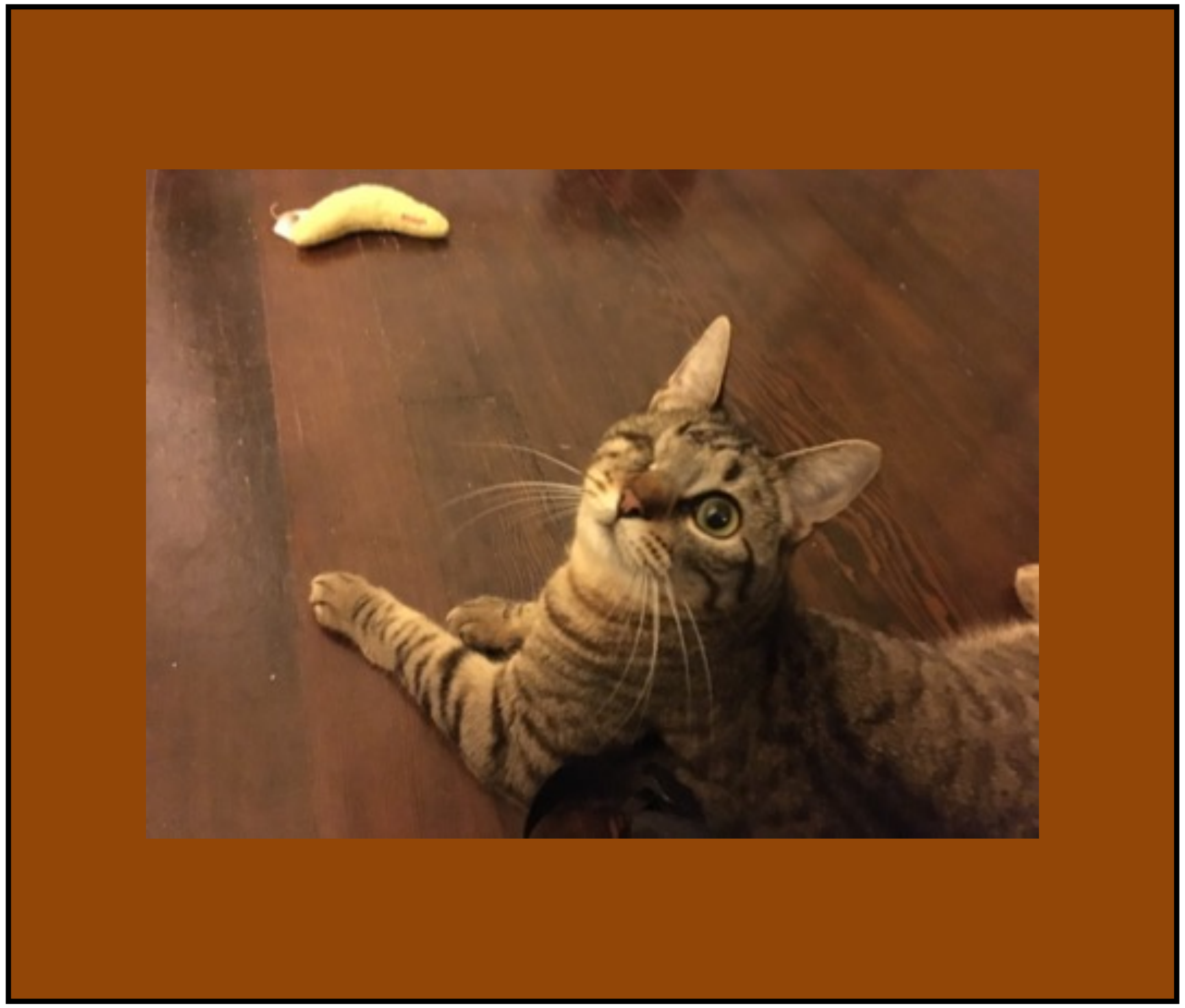

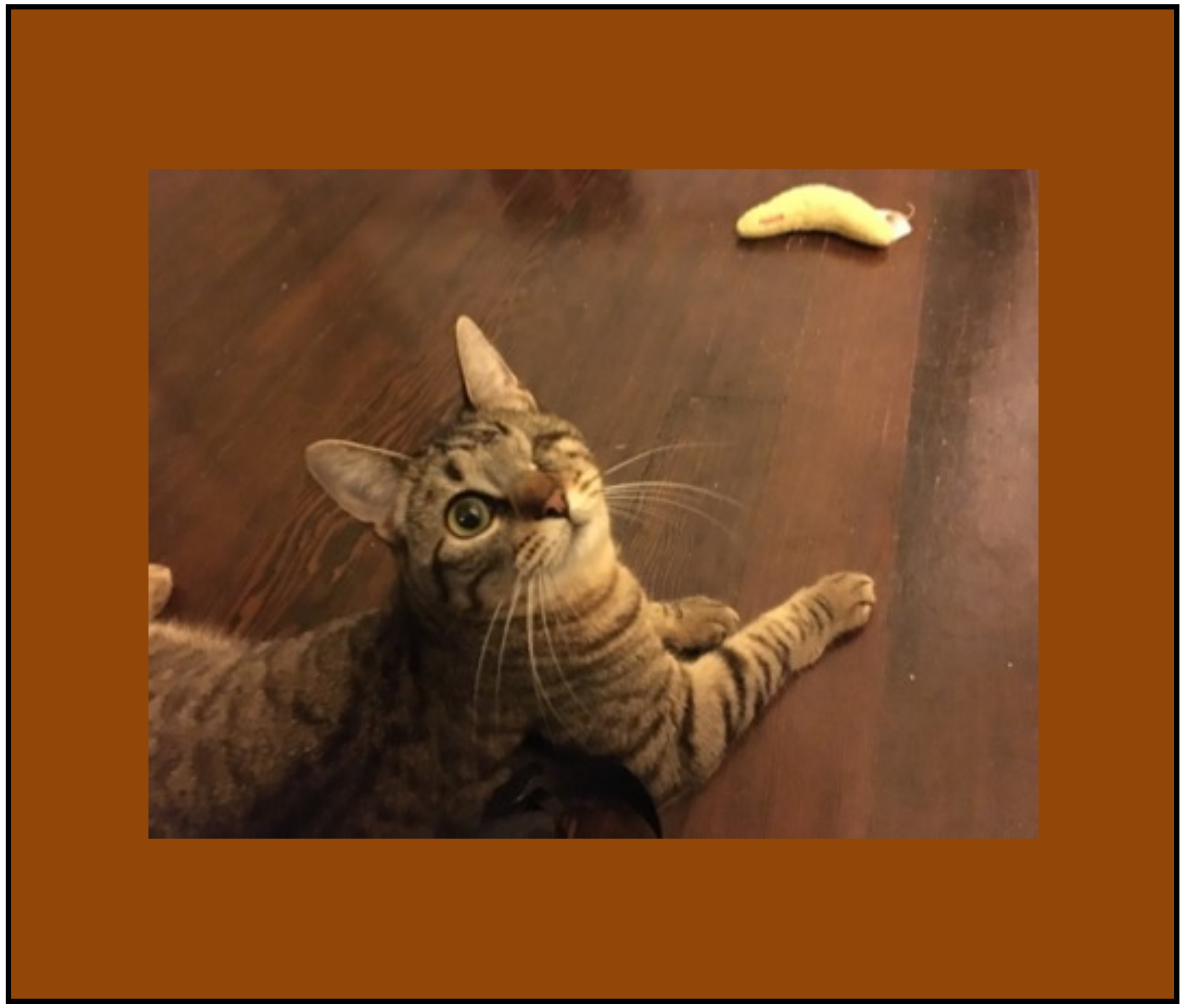

Mirror symmetry: $\{e\,, M\}$, with $Mx(u_1, u_2) = x(-u_1, u_2)$.

Rigid transformation symmetries

- We can combine all these transformations into a single group, the Affine Group $\mathrm{Aff}(\mathbb{R}^2)$.

- It has 6 degrees of freedom; in the representation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

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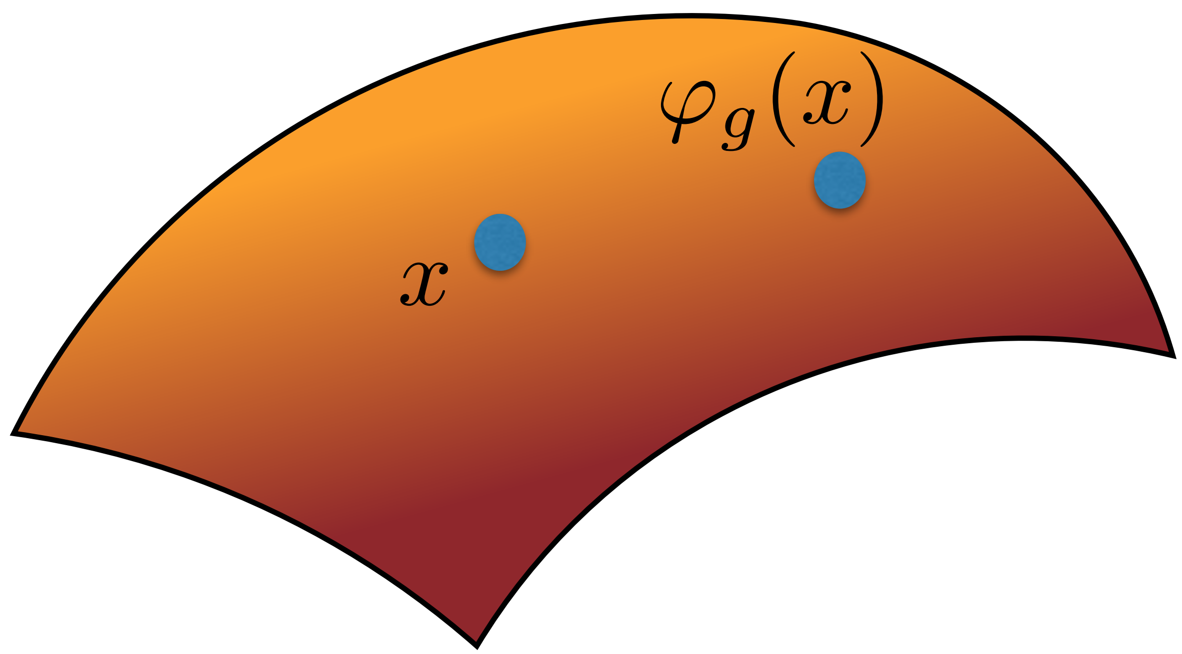
- For some groups, we might only observe partial invariance (e.g. rotation and dilation).
- In speech, the underlying group modeling time-frequency shifts is the *Heisenberg* group.

Invariant Representations

- Given a transformation group G and an input x, the *action* of G onto x is called an *orbit*:

$$G \cdot x = \{\varphi_g(x), g \in G\}$$



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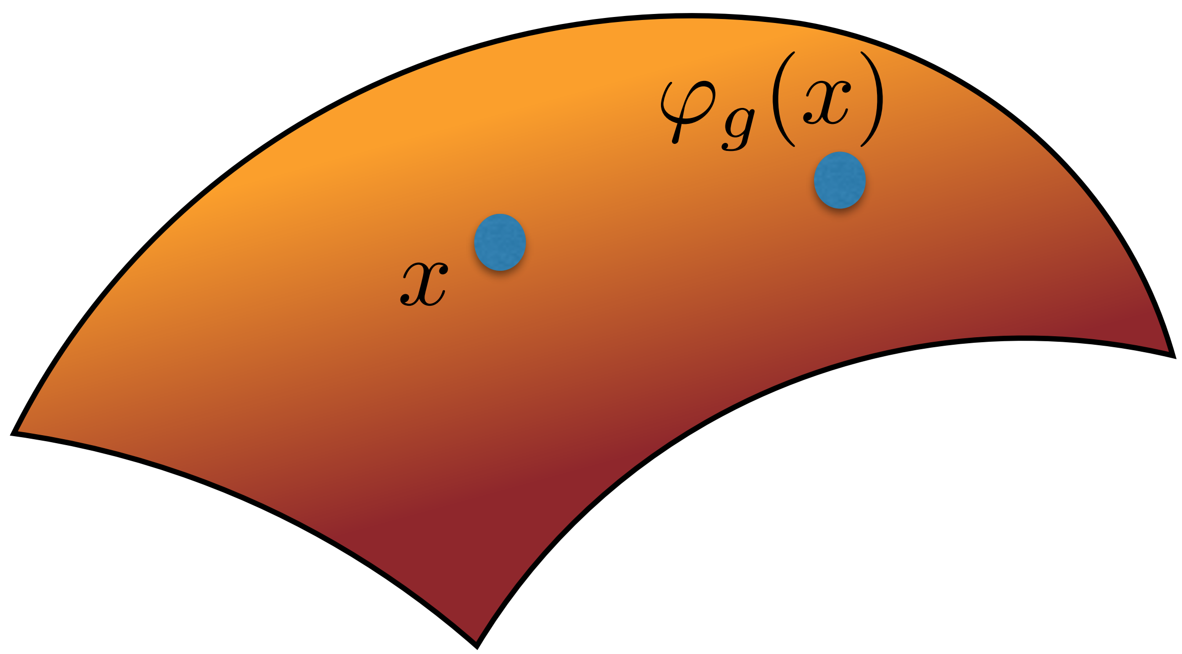

- Impact on the learning task?
- Since our estimator is linear in $\Phi(x)$, $\Phi(G \cdot x)$ should be "flat".

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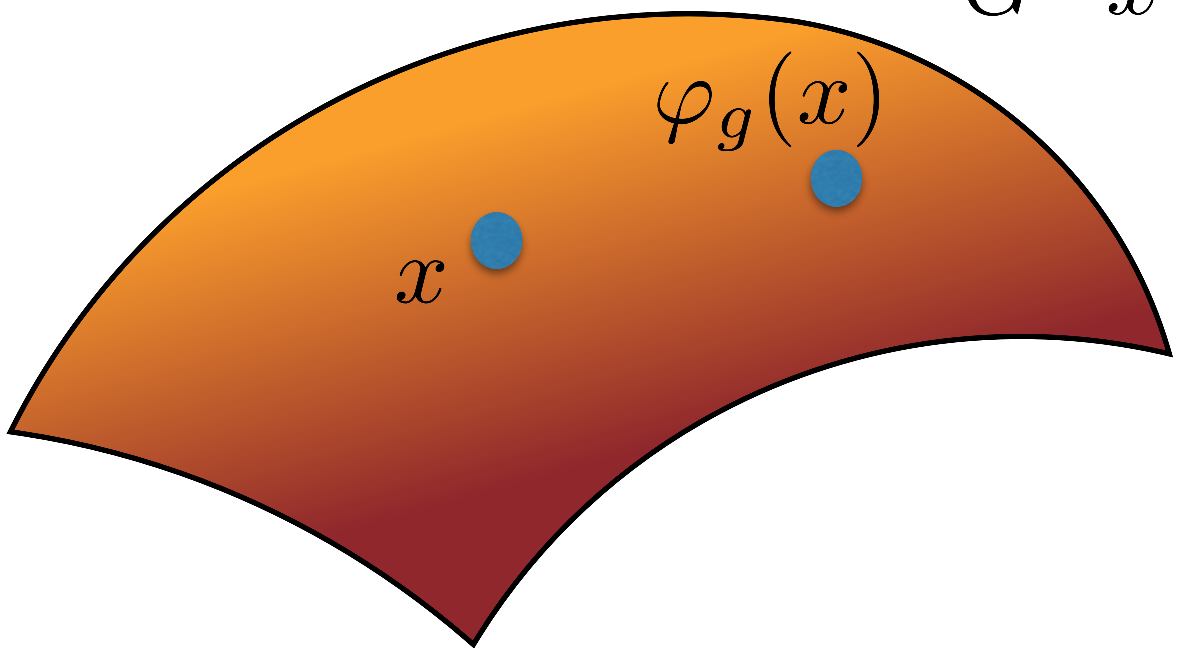

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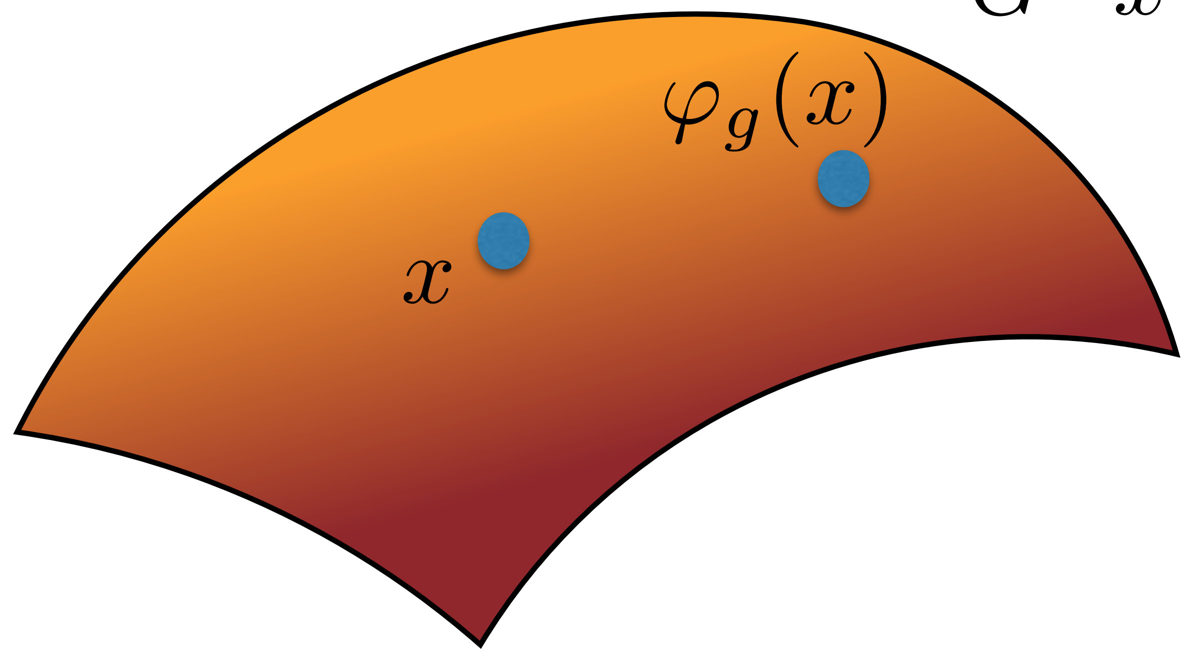

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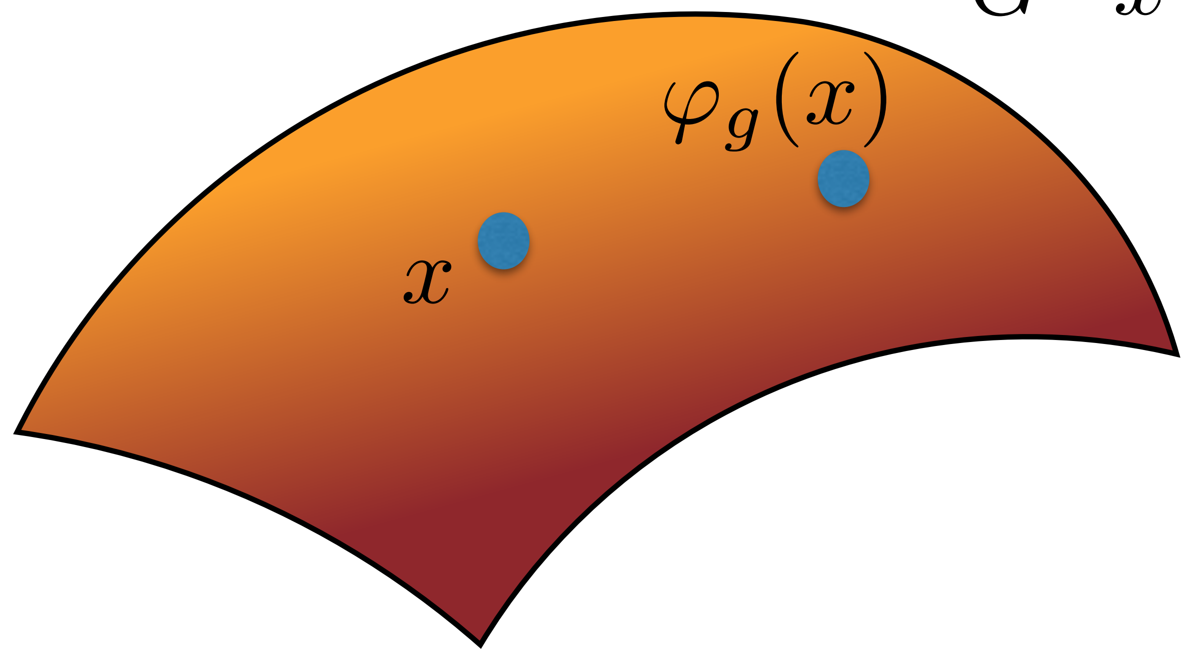

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- Group symmetries are not sufficient to beat the curse of dimensionality.

From Invariance to Stability

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- Although image and audio recognition does not have high-dimensional symmetry groups, it is *stable* to local deformations.

$$x \in L^2(\mathbb{R}^m) \ , \ \tau : \mathbb{R}^m \to \mathbb{R}^m \text{ diffeomorphism}$$

$$x_\tau = \varphi_\tau(x) \ , \ x_\tau(u) = x(u - \tau(u))$$

φ_τ is a change of variables: (think of x_τ as adding noise to the pixel *locations* rather than to the pixel values)

From Invariance to Stability

   

- Informally, if $\|\tau\|$ measures the amount of deformation, many recognition tasks satisfy

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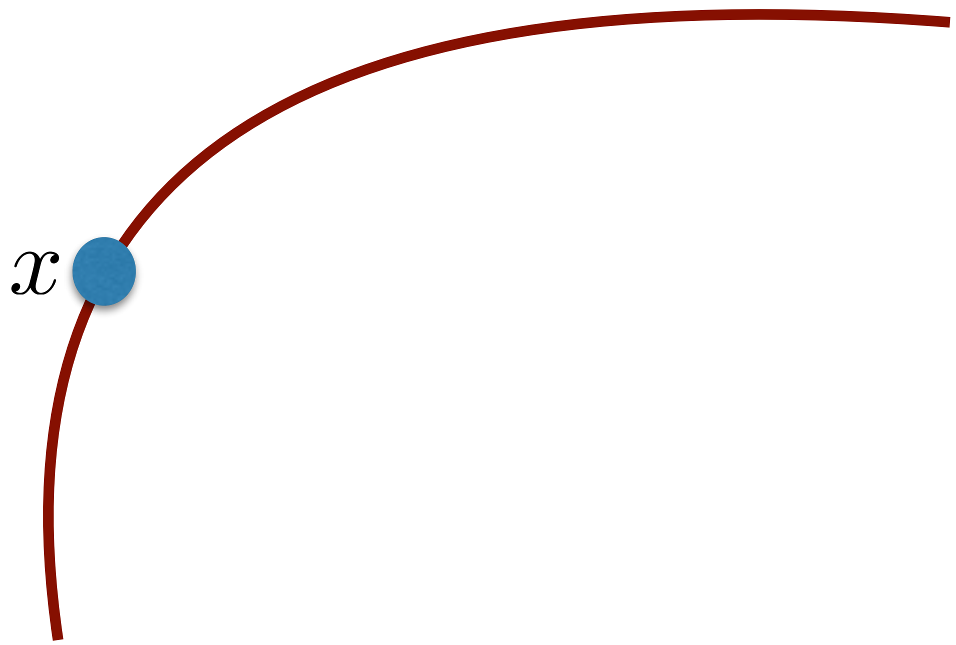
$$\forall\ x, \tau,\ \ |f(x) - f(x_\tau)| \lesssim \|\tau\|$$

- If our representation is stable, then

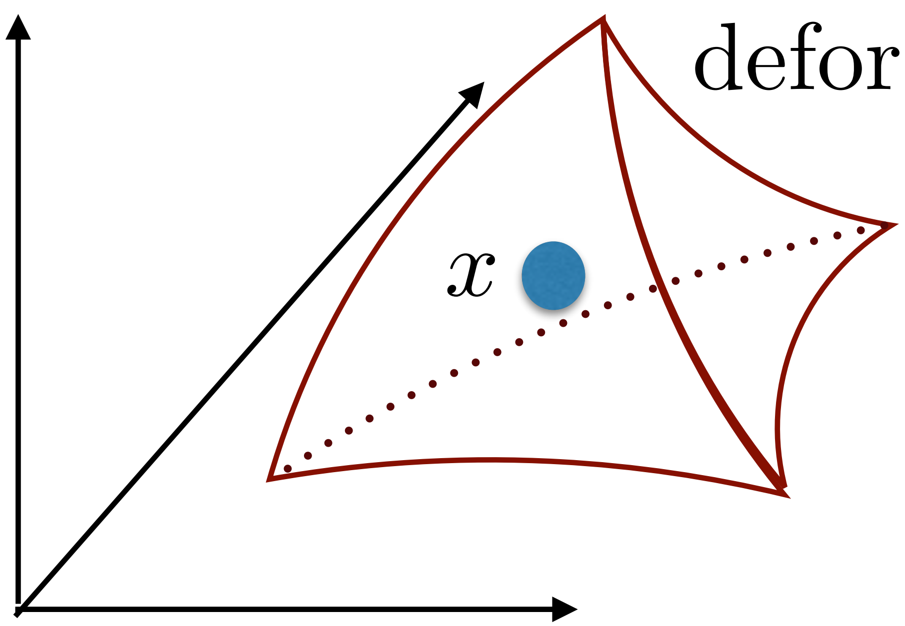
$$\forall\ x, \tau,\ \ \|\Phi(x) - \Phi(x_\tau)\| \leq C\|\tau\| \Longrightarrow |\hat{f}(x) - \hat{f}(x_\tau)| \leq \tilde{C}\|\tau\|$$

Filling the space with deformations

symmetry group: low dimension



deformations fill the space



Deformations in Image/Audio Recognition

- Can model 3D viewpoint changes, changes in pitch/ timbre in speech recognition.

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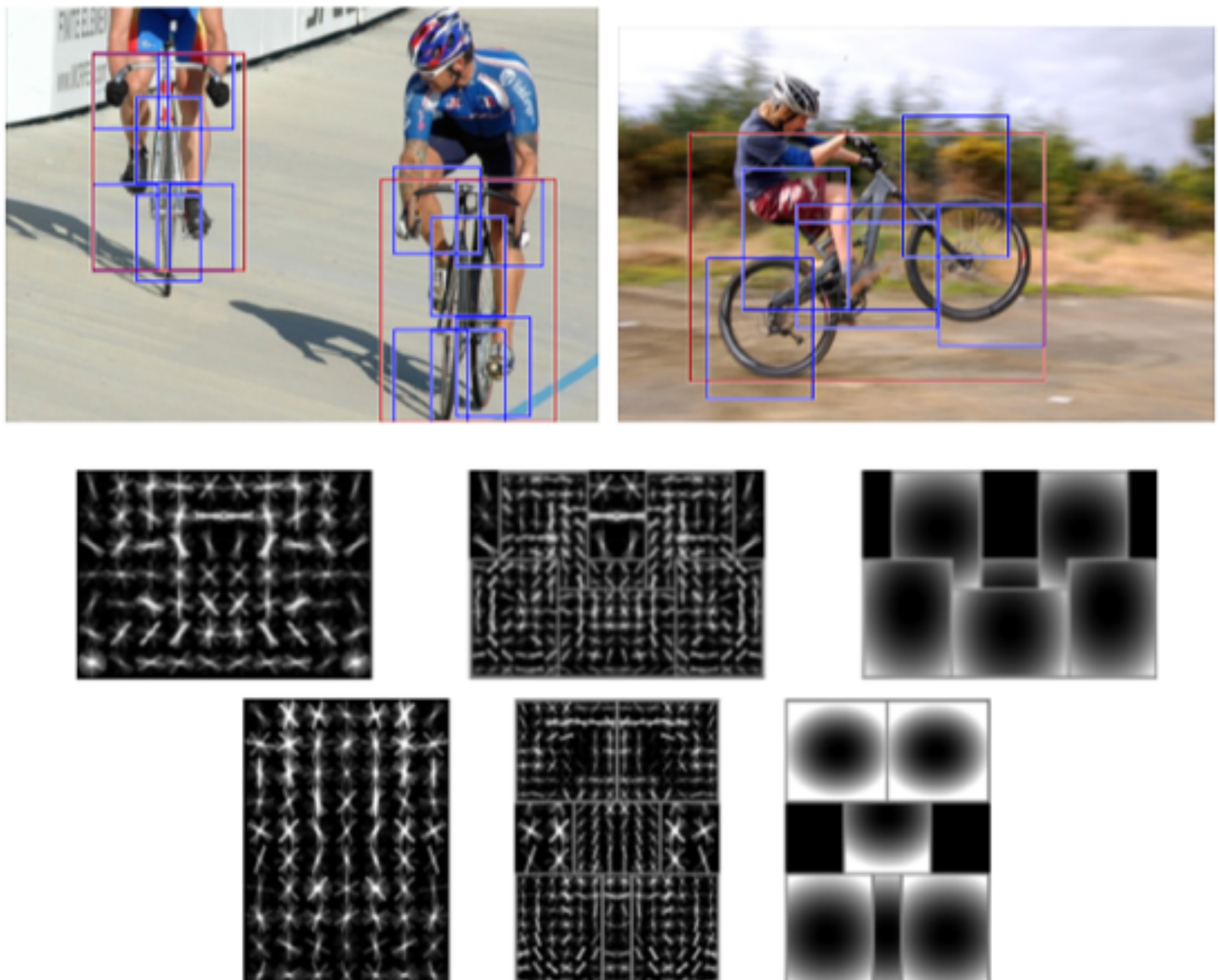
- Can model 3D viewpoint changes, changes in pitch/timbre in speech recognition.
- Deformable parts model [Feltzenszwalb et al, '10]



- State-of-the-art on object detection pre-CNN.

Deformations in Image/Audio Recognition

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- Deformable templates [Grenader, Younes, Trouvé, Amit et al.]
  - Equip deformable templates with differentiable structure

Deformations in Image/Audio Recognition

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- Data augmentation in Object classification
  - Mostly rigid transformations (random shifts, flips).

Stability Condition

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- If we fix the 'template' *x* and consider the mapping

$$F\ :\ \tau \mapsto \Phi(x_\tau)$$

the previous condition becomes

$$\|F(\tau) - F(0)\| \leq C\|\tau\|\ ,$$

thus *F* is Lipschitz with respect to the deformation metric $\|\tau\|$ *uniformly* on x .

Stationarity Prior

- Two clips. Goal: distinguish which is which.

clip1 clip2 clip ?

Stationarity Prior

- Same experiment. Goal: distinguish which is which.

clip3 clip4 clip ?

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- Typically, the latter is harder. Reasons?

"Summary Statistics in auditory perception", McDermott & Simoncelli, Nature Neurosc.'13

Stationarity Prior

- Same experiment. Goal: distinguish which is which.

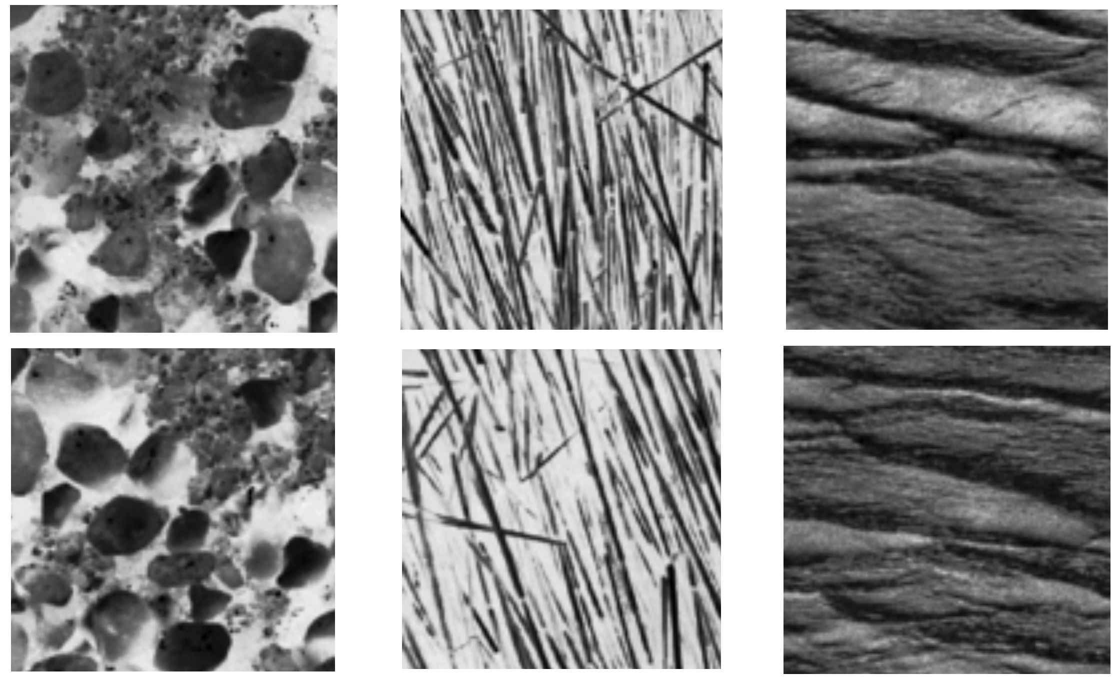
clip3 clip4 clip ?

- Typically, the latter is harder. Reasons?
- Despite having more information, the discrimination is worse because we construct temporal averages in presence of *stationary* inputs.

"Summary Statistics in auditory perception", McDermott & Simoncelli, Nature Neurosc.'13

Representation of Stationary Processes

$x(u)$: realizations of a stationary process $X(u)$ (not Gaussian)



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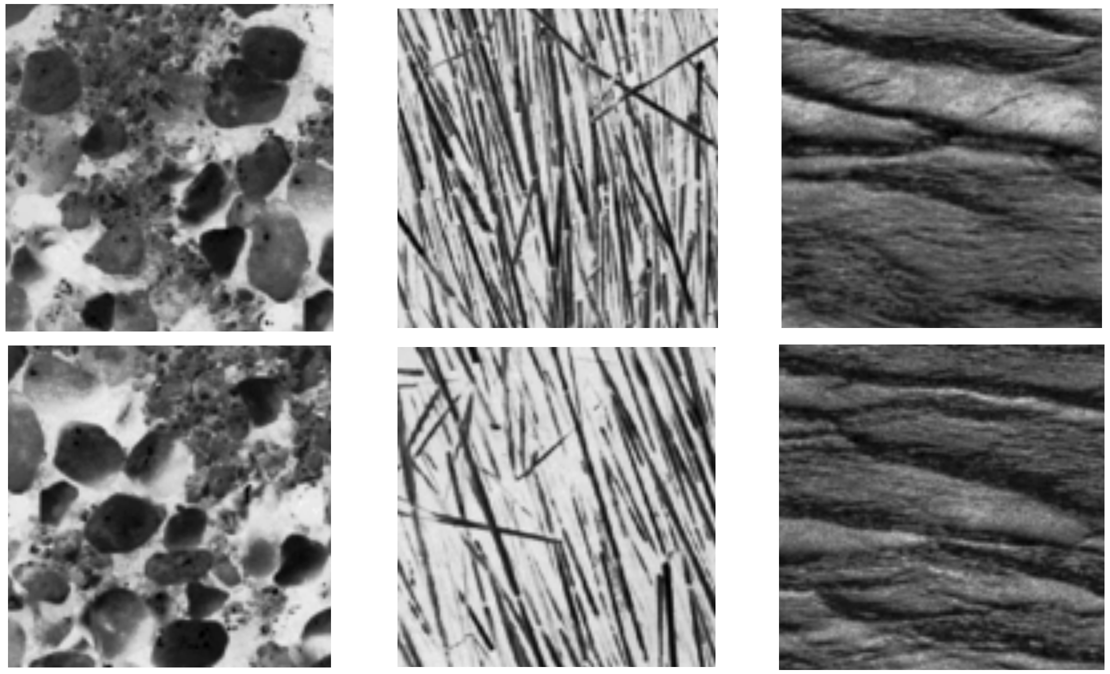

$$\Phi(X) = \{E(f_i(X))\}_i$$

Estimation from samples $x(n)$: $\widehat{\Phi}(X) = \left\{ \frac{1}{N} \sum_n f_i(x)(n) \right\}_i$

Discriminability: need to capture high-order moments
Stability: $E(\|\widehat{\Phi}(X) - \Phi(X)\|^2)$ small

Ergodicity

- Which class of processes satisfy the following?

$$\forall\ i\ ,\ \frac{1}{N}\sum_{n} f_i(x)(n) \rightarrow \mathbf{E}(f_i(X)) \quad (N \rightarrow \infty)$$

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- Which class of processes satisfy the following?

$$\forall\ i\ ,\ \frac{1}{N}\sum_{n} f_i(x)(n) \to \mathbf{E}(f_i(X)) \quad (N \to \infty)$$

- These are called *ergodic* processes.
  - In statistical physics, a process with an *Integral Scale* is ergodic.
  - In statistics, *linear processes* are ergodic (provided the moments are finite).

Class-specific variability

- Besides deformations and stationary variability, object recognition is exposed to much more complex variability:






- clutter
- class-specific diversity